Provide an LTL formula over $AP = \{a, b\}$ for ...

- set of all words $A_0\, A_1\, A_2 \ldots \in \left(2^{AP}\right)^{\omega}$ such that:

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- set of all words of the form

$$\{b\}^{n_1} \, \{a\} \, \{b\}^{n_2} \, \{a\} \, \{b\}^{n_3} \, \{a\} \ldots$$

where $n_1, n_2, n_3, \ldots \geq 0$

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$$\widehat{=}\ \ \mathit{Words}(\ \Box((b \wedge \neg a)\ \mathsf{U}\ (a \wedge \neg b))\ )$$

Equivalence of LTL formulas

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Examples:

$$\varphi_1 \vee \varphi_2 \equiv \varphi_2 \vee \varphi_1$$
$$\neg\neg\varphi \equiv \varphi$$
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all equivalences
from propositional logic

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all equivalences from propositional logic

Self-duality of the next operator

$$\varphi_1 \equiv \varphi_2 \quad \text{iff} \quad \textit{Words}(\varphi_1) = \textit{Words}(\varphi_2)$$

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$$\text{iff} \quad A_0\, A_1\, A_2\, A_3 \ldots \not\models \bigcirc\varphi$$

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$$\begin{array}{lrcl}
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\text{iff} & A_1\,A_2\,A_3\ldots & \not\models & \varphi \\
\text{iff} & A_1\,A_2\,A_3\ldots & \models & \neg\varphi
\end{array}$$

Self-duality of the next operator

$$\varphi_1 \equiv \varphi_2 \quad \text{iff} \quad Words(\varphi_1) = Words(\varphi_2)$$

Claim: $\neg\bigcirc\varphi \equiv \bigcirc\neg\varphi$ "self-duality of next"

Proof:

$$
\begin{array}{lrcl}
 & A_0\, A_1\, A_2\, A_3 \ldots & \models & \neg\bigcirc\varphi \\
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\text{iff} & A_1\, A_2\, A_3 \ldots & \not\models & \varphi \\
\text{iff} & A_1\, A_2\, A_3 \ldots & \models & \neg\varphi \\
\text{iff} & A_0\, A_1\, A_2\, A_3 \ldots & \models & \bigcirc\neg\varphi
\end{array}
$$

Correct or wrong?

$$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$$

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Correct or wrong?

$$\Diamond(\varphi \lor \psi) \;\equiv\; \Diamond\varphi \lor \Diamond\psi$$

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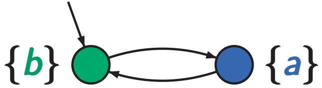
$$\Diamond(\varphi \land \psi) \;\equiv\; \Diamond\varphi \land \Diamond\psi$$

wrong,
e.g.,



$\models \Diamond b \land \Diamond a$

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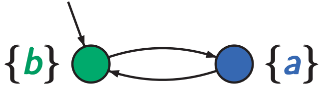

$\models \; \Diamond b \wedge \Diamond a$

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---

similarly: $\Box(\varphi \wedge \psi) \;\equiv\; \Box\varphi \wedge \Box\psi$

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Correct or wrong?

$$\Diamond\Diamond\varphi \equiv \Diamond\varphi$$

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correct Analogous: $\Box\Box\varphi \equiv \Box\varphi$

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$\bigcirc\Box\varphi \equiv \Box\bigcirc\varphi \stackrel{\text{def}}{=} \psi$

correct

note that:

$A_0\, A_1\, A_2 \ldots \models \psi$ iff $A_i\, A_{i+1} \ldots \models \varphi$ for all $i \geq 1$

Correct or wrong?

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$\Diamond\Box\varphi \equiv \Box\Diamond\varphi$

$\Box\Diamond \;\widehat{=}\;$ infinitely often

$\Diamond\Box \;\widehat{=}\;$ eventually forever

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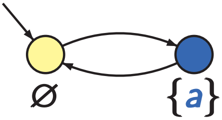

$\models \Box\Diamond a$

$\not\models \Diamond\Box a$

Expansion laws

Expansion law for U

until: $\varphi \, \mathsf{U} \, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

Expansion laws for U and $\Diamond$

until: $\varphi \,\mathsf{U}\, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$

eventually: $\Diamond\psi \equiv \psi \vee \bigcirc\Diamond\psi$

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note: $\Diamond\psi = \mathit{true} \,\mathsf{U}\, \psi$

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$$\equiv \psi \vee \bigcirc \Diamond \psi$$

Expansion laws for $\mathbf{U}$, $\Diamond$ and $\Box$

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always: $\Box \psi \equiv$ **?**

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$$\Box \psi = \neg \Diamond \neg \psi$$

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$$\Box\psi = \neg\Diamond\neg\psi$$

$$\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \longleftarrow \boxed{\text{expansion law for } \Diamond}$$

Expansion laws for U, $\Diamond$ and $\Box$

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$$\begin{aligned} \Box\psi &= \neg\Diamond\neg\psi \\ &\equiv \neg(\neg\psi \vee \bigcirc\Diamond\neg\psi) \\ &\equiv \neg\neg\psi \wedge \neg\bigcirc\Diamond\neg\psi \quad \longleftarrow \boxed{\text{de Morgan}} \end{aligned}$$

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&\equiv \psi \wedge \neg \bigcirc \Diamond \neg \psi \quad \longleftarrow \boxed{\text{double negation}}
\end{aligned}
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$\longleftarrow$ self duality of $\bigcirc$

Expansion laws for U, $\Diamond$ and $\Box$

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&\equiv \psi \wedge \bigcirc\neg\Diamond\neg\psi \\
&\equiv \psi \wedge \bigcirc\Box\psi
\end{aligned}
$$

$\longleftarrow$ definition of $\Box$

Expansion laws are fixed point equations

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Expansion laws are fixed point equations

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eventually: $\boxed{\Diamond \psi} \equiv \psi \vee \bigcirc \boxed{\Diamond \psi}$

always: $\boxed{\Box \psi} \equiv \psi \wedge \bigcirc \boxed{\Box \psi}$

Expansion laws are fixed point equations

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eventually: $\Diamond \psi \equiv \psi \vee \bigcirc \Diamond \psi$

always: $\Box \psi \equiv \psi \wedge \bigcirc \Box \psi$

...don't yield a complete characterization, e.g.,

$$\mathit{false} \equiv a \wedge \bigcirc \mathit{false}$$
$$\Box a \equiv a \wedge \bigcirc \Box a$$

consider $\psi = a$

Expansion laws are fixed point equations

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...don't yield a complete characterization, e.g.,

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$$\Box a \equiv a \wedge \bigcirc \Box a$$

although

$$\Box a \not\equiv false$$

Expansion laws are fixed point equations

until: $\varphi \, \mathsf{U} \, \psi \;\equiv\; \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$

least fixed point

eventually: $\Diamond\psi \;\equiv\; \psi \;\vee\; \bigcirc\Diamond\psi$

least fixed point

always: $\Box\psi \;\equiv\; \psi \;\wedge\; \bigcirc\Box\psi$

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although
$\Box a \;\not\equiv\; false$

Expansion laws are fixed point equations

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least fixed point

always: $\Box\psi \equiv \psi \wedge \bigcirc\Box\psi$

greatest fixed point

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$$\Box a \equiv a \wedge \bigcirc\Box a$$

although $\Box a \not\equiv false$

Expansion law for U

The LTL formula $\chi = \varphi \mathbf{U} \psi$ is the least solution of

$$\chi \equiv \psi \vee (\varphi \wedge \bigcirc \chi)$$

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i.e., $\mathit{Words}(\varphi \mathbf{U} \psi)$ least LT-property E s.t.

$$E = \mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) : A_1 A_2 \ldots \in E\}$$

Expansion law for U

The LTL formula $\chi = \varphi \, \mathbf{U} \, \psi$ is the least solution of

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i.e., $\mathit{Words}(\varphi \, \mathbf{U} \, \psi)$ least LT-property E s.t.

$$E = \mathit{Words}(\psi) \cup \{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) : A_1 A_2 \ldots \in E\}$$

It even holds that $\mathit{Words}(\varphi \, \mathbf{U} \, \psi)$ least LT-property E s.t.

(1) $\mathit{Words}(\psi) \subseteq E$

(2) $\{A_0 A_1 A_2 \ldots \in \mathit{Words}(\varphi) : \ A_1 A_2 \ldots \in E\} \subseteq E$

The weak until operator W

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deriving "always" and "until" from "weak until":

$$\Box\varphi \equiv \, ?$$

The weak until operator W

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$$\varphi \, \mathsf{U} \, \psi \equiv \, ?$$

The weak until operator W

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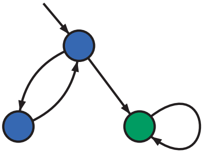
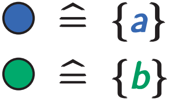
$$\Box\varphi \equiv \varphi \, \mathsf{W} \, \mathit{false}$$

$$\varphi \, \mathsf{U} \, \psi \equiv (\varphi \, \mathsf{W} \, \psi) \wedge \Diamond\psi$$

Does $\mathcal{T} \models a \mathsf{W} b$ hold?





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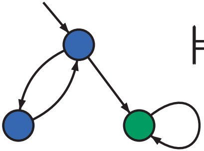
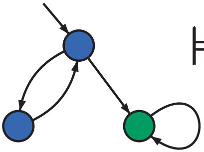


$\models a \mathsf{W}\, b$

 $\widehat{=}$ $\{a\}$

 $\widehat{=}$ $\{b\}$

Does $\mathcal{T} \models a \, \mathsf{W} \, b$ hold?

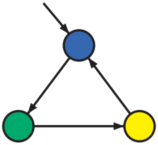


$\models a \, \mathsf{W} \, b$

blue $\widehat{=} \{a\}$

green $\widehat{=} \{b\}$

yellow $\widehat{=} \varnothing$



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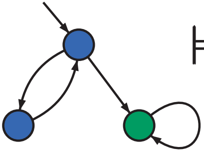
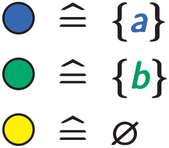
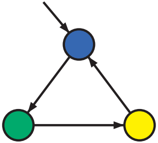


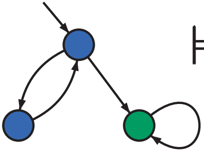
$\models a \, \mathsf{W} \, b$



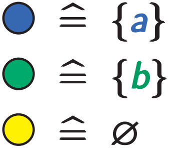


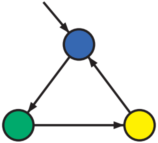
$\models \; a \, \mathsf{W} \, b \quad$ (even $a \, \mathsf{U} \, b$)

Does $\mathcal{T} \models a \,\mathsf{W}\, b$ hold?

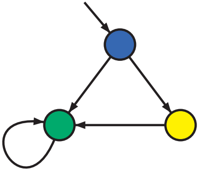


$\models a \,\mathsf{W}\, b$

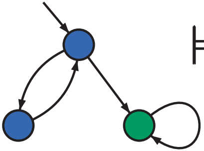




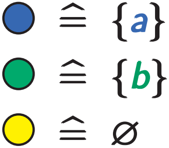
$\models a \,\mathsf{W}\, b$ (even $a \,\mathsf{U}\, b$)

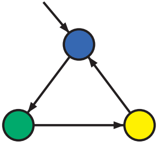


Does $\mathcal{T} \models a \, \mathsf{W} \, b$ hold?

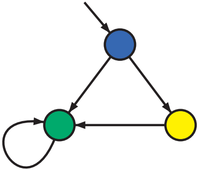


$\models a \, \mathsf{W} \, b$



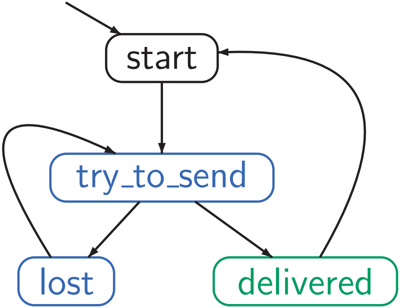


$\models a \, \mathsf{W} \, b$ (even $a \, \mathsf{U} \, b$)

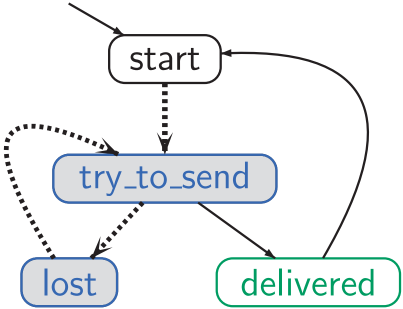


$\not\models a \, \mathsf{W} \, b$

Example: simple communication protocol

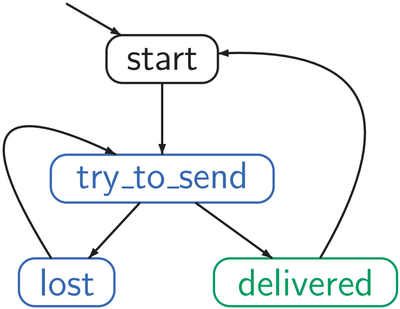


Example: simple communication protocol



$$\mathcal{T} \not\models \Box(\textit{blue} \longrightarrow \textit{blue}\ \mathsf{U}\ \textit{delivered})$$

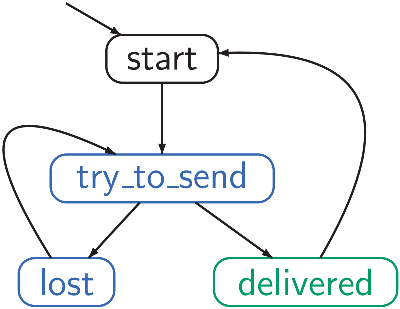
Example: until versus weak until



$$\mathcal{T} \not\models \Box(\mathit{blue} \longrightarrow \mathit{blue}\ \mathsf{U}\ \mathit{delivered})$$

$$\mathcal{T} \models \Box(\mathit{blue} \longrightarrow \mathit{blue}\ \mathsf{W}\ \mathit{delivered})$$

Example: until versus weak until



constrained liveness:

$$\mathcal{T} \not\models \Box(\textit{blue} \longrightarrow \textit{blue}\ \mathsf{U}\ \textit{delivered})$$

safety:

$$\mathcal{T} \models \Box(\textit{blue} \longrightarrow \textit{blue}\ \mathsf{W}\ \textit{delivered})$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \overset{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

goal: express $\neg(\varphi \, \mathsf{U} \, \psi)$ via **W**, and vice versa

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \overset{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \lor \Box\varphi$$

$$\neg(\varphi \, \mathsf{U} \, \psi)$$

$$\equiv ((\varphi \land \neg\psi) \, \mathsf{U} \, (\neg\varphi \land \neg\psi)) \lor \Box(\varphi \land \neg\psi)$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box \varphi$$

$$\begin{aligned} & \neg(\varphi \, \mathsf{U} \, \psi) \\ \equiv \;& ((\varphi \wedge \neg \psi) \; \mathsf{U} \; (\neg \varphi \wedge \neg \psi)) \vee \Box(\varphi \wedge \neg \psi) \\ \equiv \;& (\varphi \wedge \neg \psi) \, \mathsf{W} \, (\neg \varphi \wedge \neg \psi) \end{aligned}$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

$$\begin{aligned}
& \neg(\varphi \, \mathsf{U} \, \psi) \\
\equiv \; & ((\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv \; & (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi) \\
\equiv \; & (\neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)
\end{aligned}$$

Duality of U and W

$$\varphi \,\mathsf{W}\, \psi \;\stackrel{\text{def}}{=}\; (\varphi \,\mathsf{U}\, \psi) \vee \Box\varphi$$

$$\begin{aligned}
& \neg(\varphi \,\mathsf{U}\, \psi) \\
\equiv\; & ((\varphi \wedge \neg\psi) \,\mathsf{U}\, (\neg\varphi \wedge \neg\psi)) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv\; & (\varphi \wedge \neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi) \\
\equiv\; & (\neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi)
\end{aligned}$$

$$\neg(\varphi \,\mathsf{U}\, \psi) \equiv (\neg\psi) \,\mathsf{W}\, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \,\mathsf{W}\, \psi) \equiv \;?$$

Duality of U and W

$$\varphi \, \mathsf{W} \, \psi \;\stackrel{\text{def}}{=}\; (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$$

$$\begin{aligned}
& \neg(\varphi \, \mathsf{U} \, \psi) \\
\equiv \;& \big((\varphi \wedge \neg\psi) \; \mathsf{U} \; (\neg\varphi \wedge \neg\psi)\big) \vee \Box(\varphi \wedge \neg\psi) \\
\equiv \;& (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi) \\
\equiv \;& (\neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)
\end{aligned}$$

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

Expansion laws for U and W

Expansion laws for U and W

$$\varphi \, \mathsf{U} \, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$$

$$\varphi \, \mathsf{W} \, \psi \quad \equiv \quad ?$$

Expansion laws for U and W

$$\varphi \, \mathsf{U} \, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$$

$$\varphi \, \mathsf{W} \, \psi \quad \equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$$

Expansion laws for U and W

$$\varphi \,\mathsf{U}\, \psi \equiv \psi \lor (\varphi \land \bigcirc(\varphi \,\mathsf{U}\, \psi))$$

smallest solution

$$\varphi \,\mathsf{W}\, \psi \equiv \psi \lor (\varphi \land \bigcirc(\varphi \,\mathsf{W}\, \psi))$$

Expansion laws for U and W

$\varphi \,\mathsf{U}\, \psi$	$\equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$	smallest solution
$\varphi \,\mathsf{W}\, \psi$	$\equiv \quad \psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{W}\, \psi))$	largest solution

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi$	$\equiv$	$\psi \lor (\varphi \land \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
$\varphi \, \mathsf{W} \, \psi$	$\equiv$	$\psi \lor (\varphi \land \bigcirc(\varphi \, \mathsf{W} \, \psi))$	largest solution

Words$(\varphi \, \mathsf{U} \, \psi)$ smallest LT-property $\boldsymbol{E}$ s.t.

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
$\varphi \, \mathsf{W} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$	largest solution

$Words(\varphi \, \mathsf{U} \, \psi)$ smallest LT-property E s.t.

(1) $Words(\psi) \subseteq E$

(2) $\{A_0 A_1 A_2 \ldots \in Words(\varphi) : A_1 A_2 \ldots \in E\} \subseteq E$

Expansion laws for U and W

$\varphi \mathbf{U} \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{U} \psi))$	smallest solution
$\varphi \mathbf{W} \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \mathbf{W} \psi))$	largest solution

$Words(\varphi \mathbf{U} \psi)$ smallest LT-property E s.t.

(1) $Words(\psi) \subseteq E$

(2) $\{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\} \subseteq E$

$$\Updownarrow$$

$$Words(\psi) \cup \{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
$\varphi \, \mathsf{W} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$	largest solution

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Expansion laws for U and W

$\varphi \,\mathsf{U}\, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$	smallest solution
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$Words(\varphi \,\mathsf{U}\, \psi)$ smallest LT-property E s.t.

$$Words(\psi) \cup \{A_0A_1A_2\ldots \in Words(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

$Words(\varphi \,\mathsf{W}\, \psi)$ largest LT-property E s.t.

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
$\varphi \, \mathsf{W} \, \psi$	$\equiv$	$\psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$	largest solution

$\mathit{Words}(\varphi \, \mathsf{U} \, \psi)$ smallest LT-property E s.t.

$$\mathit{Words}(\psi) \cup \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

$\mathit{Words}(\varphi \, \mathsf{W} \, \psi)$ largest LT-property E s.t.

$$\mathit{Words}(\psi) \cup \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\} \supseteq E$$

Expansion laws for U and W

$\varphi \, \mathsf{U} \, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \, \psi))$	smallest solution
$\varphi \, \mathsf{W} \, \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{W} \, \psi))$	largest solution

$\mathit{Words}(\varphi \, \mathsf{U} \, \psi)$ smallest LT-property E s.t.

$$\mathit{Words}(\psi) \cup \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\} \subseteq E$$

$\mathit{Words}(\varphi \, \mathsf{W} \, \psi)$ largest LT-property E s.t.

$$E \subseteq \mathit{Words}(\psi) \cup \{A_0A_1A_2\ldots \in \mathit{Words}(\varphi) : A_1A_2\ldots \in E\}$$

Expansion laws for U and W

$$\varphi \,\mathsf{U}\, \psi \quad \equiv \quad \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{U}\, \psi))$$

smallest solution

$$\varphi \,\mathsf{W}\, \psi \quad \equiv \quad \psi \;\vee\; (\varphi \wedge \bigcirc(\varphi \,\mathsf{W}\, \psi))$$

largest solution

Expansion laws for U, W, $\Diamond$, and $\Box$

$$\varphi \, \mathsf{U} \, \psi \quad \equiv \quad \psi \ \lor \ (\varphi \land \bigcirc(\varphi \, \mathsf{U} \, \psi))$$

smallest solution

$$\Diamond\psi \quad \equiv \quad \psi \ \lor \ \bigcirc\Diamond\psi$$

smallest solution

$$\varphi \, \mathsf{W} \, \psi \quad \equiv \quad \psi \ \lor \ (\varphi \land \bigcirc(\varphi \, \mathsf{W} \, \psi))$$

largest solution

$$\Box\varphi \quad \equiv \quad \varphi \land \bigcirc\Box\varphi$$

largest solution

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remind: $\Diamond\psi \ = \ \mathit{true} \, \mathsf{U} \, \psi$, $\quad \Box\varphi \ \equiv \ \varphi \, \mathsf{W} \, \mathit{false}$