Linear vs branching time

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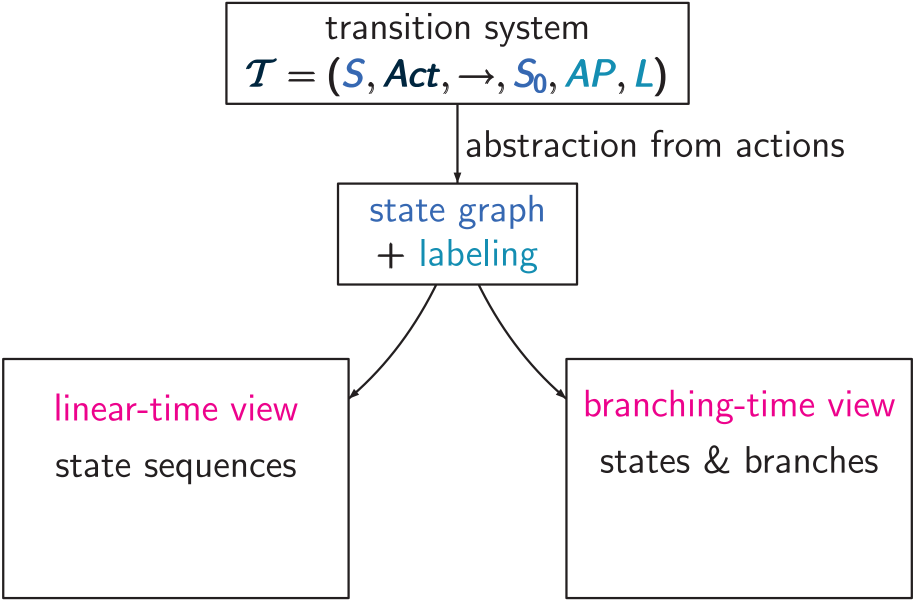

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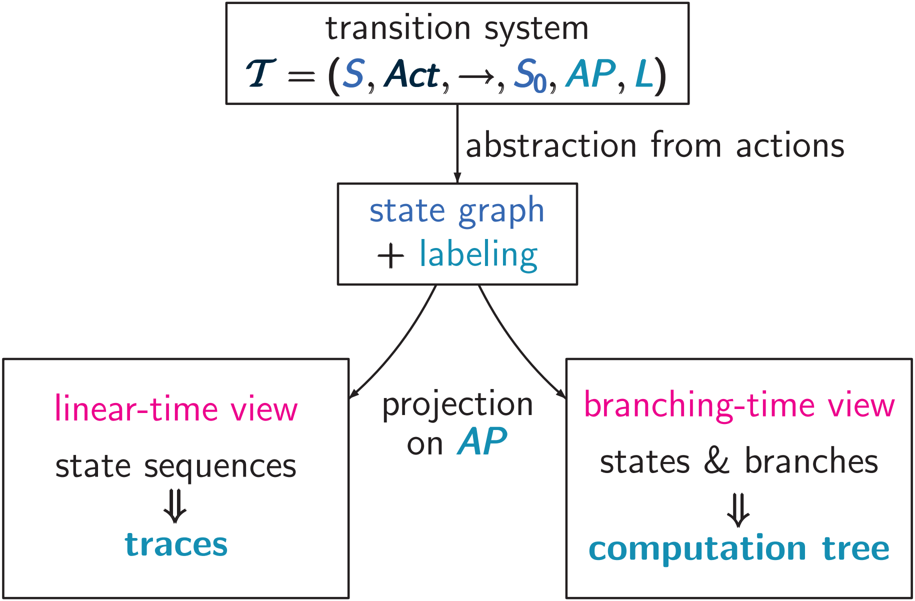

Computation tree

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The computation tree of a transition system $\mathcal{T} = (S, Act, \to, s_0, AP, L)$ arises by:

- unfolding into a tree
- abstraction from the actions
- projection of the states s to their labels $L(s) \subseteq AP$

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The computation tree of state s_0 in a transition system $\mathcal{T} = (S, Act, \rightarrow, S_0, AP, L)$ arises by:

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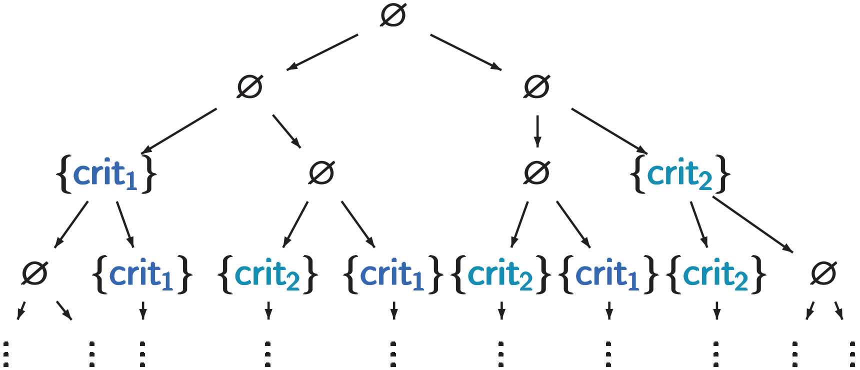
Example: computation tree

mutual exclusion with semaphore and $AP = \{crit_1, crit_2\}$:



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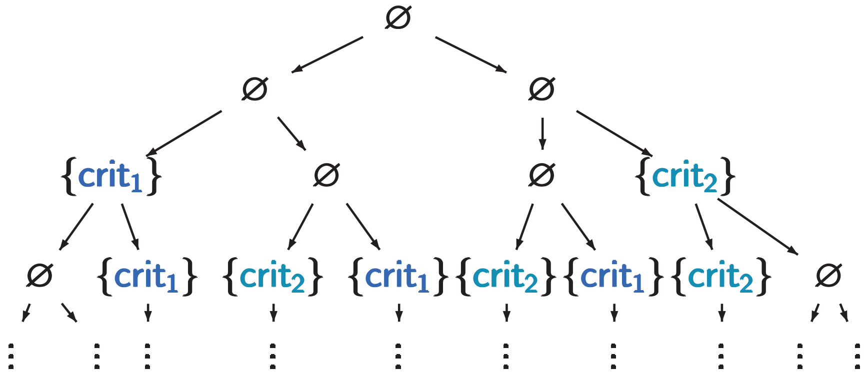

path $\langle nc_1, nc_2 \rangle$ $\langle wait_1, nc_2 \rangle$ $\langle crit_1, nc_2 \rangle$ $\langle crit_1, wait_2 \rangle$ ...

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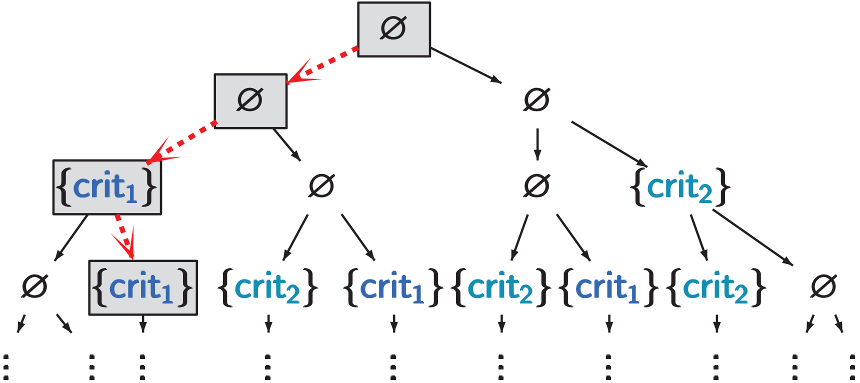

path	$\langle nc_1, nc_2 \rangle$	$\langle wait_1, nc_2 \rangle$	$\langle crit_1, nc_2 \rangle$	$\langle crit_1, wait_2 \rangle$	...
↓	↓	↓	↓	↓	
trace	$\varnothing$	$\varnothing$	$\{crit_1\}$	$\{crit_1\}$	...

Linear vs. branching time

	linear time	branching time
behavior	path based	state based

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fairness	can be encoded	requires special treatment

Computation Tree Logic (CTL)

CTL (state) formulas:

$$\Phi \ ::= \ \mathit{true} \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \exists\varphi \mid \forall\varphi$$

CTL path formulas:

$$\varphi \ ::= \ \bigcirc\Phi \mid \Phi_1 \,\mathsf{U}\, \Phi_2$$

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Syntax of CTL

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unconditional process fairness $\forall\Box\forall\Diamond crit_1 \wedge \forall\Box\forall\Diamond crit_2$

Example: 15-puzzle

6	8	2	12
4	1	13	5
	9	10	14
7	11	15	3

$\overset{?}{\rightsquigarrow}$

1	2	3	4
5	6	7	8
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↑

states:	game configurations
transitions:	legal moves

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- representation as parallel system:

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  with shared variables $\mathit{field}[i]$ for $i = 1, \ldots, 16$

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CTL specification:

$$\exists\Diamond \bigwedge_{1 \leq i \leq 15} \text{“}\mathit{piece}\ i \text{ on } \mathit{field}[i]\text{”}$$

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CTL specification: seeking for a **witness** for

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Semantics of CTL

define a satisfaction relation $\models$ for CTL formulas over AP and a given TS $\mathcal{T} = (S, Act, \to, S_0, AP, L)$

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- interpretation of state formulas over the states
- interpretation of path formulas over the paths (infinite path fragments)

for infinite path fragment $\pi = s_0\, s_1\, s_2\, \ldots$:

$\pi \models \mathit{true}$		
$\pi \models a$	iff	$s_0 \models a$,i.e., $a \in L(s_0)$
$\pi \models \varphi_1 \wedge \varphi_2$	iff	$\pi \models \varphi_1$ and $\pi \models \varphi_2$
$\pi \models \neg\varphi$	iff	$\pi \not\models \varphi$
$\pi \models \bigcirc\varphi$	iff	$\mathit{suffix}(\pi, 1) = s_1\, s_2\, s_3\, \ldots \models \varphi$
$\pi \models \varphi_1 \,\mathsf{U}\, \varphi_2$	iff	there exists $j \geq 0$ such that

$\mathit{suffix}(\pi, j) = s_j\, s_{j+1}\, s_{j+2}\, \ldots \models \varphi_2$ and

$\mathit{suffix}(\pi, k) = s_k\, s_{k+1}\, s_{k+2}\, \ldots \models \varphi_1$ for $0 \leq k < j$

Satisfaction relation for path formulas

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semantics of derived operators:

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semantics of derived operators:

$\pi \models \Diamond \Phi$	iff	there exists $j \geq 0$ with $s_j \models \Phi$
$\pi \models \Box \Phi$	iff	for all $j \geq 0$ we have: $s_j \models \Phi$

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Satisfaction relation for state formulas

$$
\begin{array}{lll}
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s \models \exists\varphi & \text{iff} & \text{there is a path } \pi \in \mathit{Paths}(s) \\
 & & \quad \text{s.t. } \pi \models \varphi \\
s \models \forall\varphi & \text{iff} & \text{for each path } \pi \in \mathit{Paths}(s): \\
 & & \quad \pi \models \varphi
\end{array}
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satisfaction set for state formula Φ:

$$Sat(\Phi) \stackrel{\text{def}}{=} \{s \in S : s \models \Phi\}$$

Interpretation of CTL formulas over a TS

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satisfaction of state formulas over a TS $\mathcal{T}$:

$$\mathcal{T} \models \Phi \quad \text{iff} \quad S_0 \subseteq Sat(\Phi)$$

where S_0 is the set of initial states

recall: $Sat(\Phi) \;=\; \{s \in S : s \models \Phi\}$

Interpretation of CTL formulas over a TS

satisfaction of state formulas over a TS $\mathcal{T}$:

$$\mathcal{T} \models \Phi \quad \text{iff} \quad S_0 \subseteq Sat(\Phi)$$
$$\text{iff} \quad s_0 \models \Phi \text{ for all initial states } s_0 \text{ of } \mathcal{T}$$

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Semantics of the next operator

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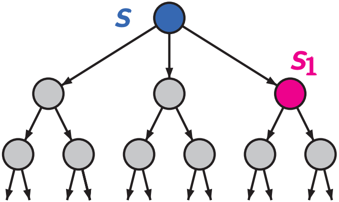
$$s \models \exists \bigcirc \Phi \quad \text{iff} \quad \text{there exists } \pi = s\, s_1\, s_2 \ldots \in Paths(s)$$
$$\text{s.t. } \pi \models \bigcirc \Phi$$

$s \models \exists\bigcirc\Phi$ iff there exists $\pi = s\, s_1\, s_2 \ldots \in \mathit{Paths}(s)$
s.t. $\pi \models \bigcirc\Phi$, i.e., $s_1 \models \Phi$

$\exists\bigcirc\Phi$



$\mathit{Post}(s) \cap \mathit{Sat}(\Phi) \neq \varnothing$

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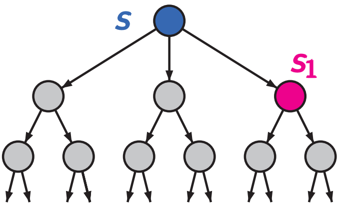
$s \models \forall\bigcirc\Phi$ iff for all $\pi = s\, s_1\, s_2 \ldots \in Paths(s)$: $\pi \models \bigcirc\Phi$

$\exists\bigcirc\Phi$



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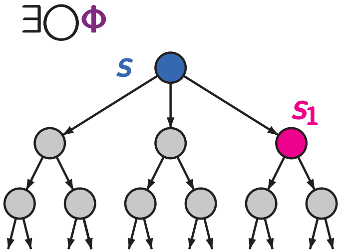

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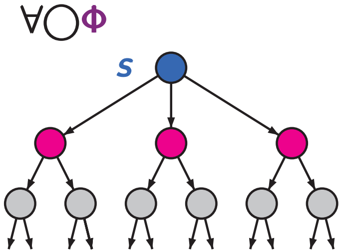
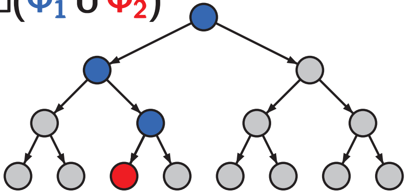


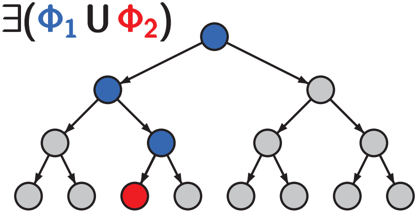
$Post(s) \subseteq Sat(\Phi)$

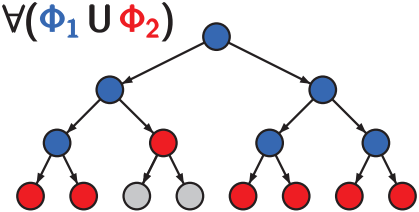
Semantics of until

$\exists(\Phi_1 \,\mathsf{U}\, \Phi_2)$



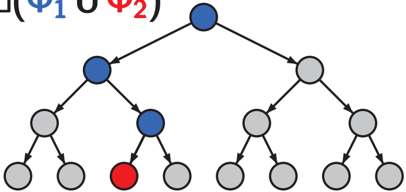
Semantics of until



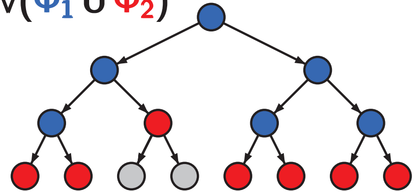


Semantics of until and eventually

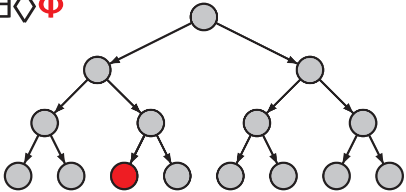
$\exists(\Phi_1 \, \mathsf{U} \, \Phi_2)$



$\forall(\Phi_1 \, \mathsf{U} \, \Phi_2)$

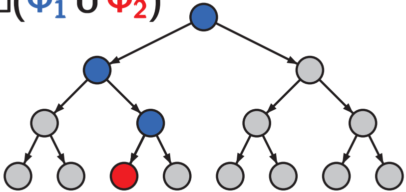


$\exists\Diamond\Phi$

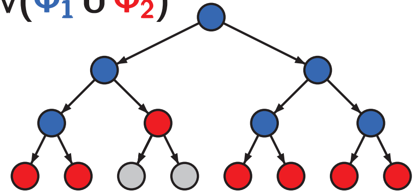


Semantics of until and eventually

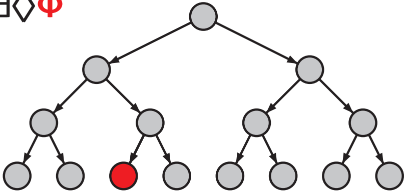
$\exists(\Phi_1 \, \mathsf{U} \, \Phi_2)$



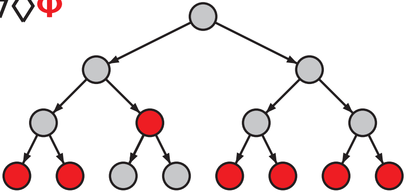
$\forall(\Phi_1 \, \mathsf{U} \, \Phi_2)$



$\exists\Diamond\Phi$

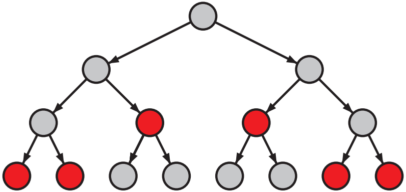


$\forall\Diamond\Phi$

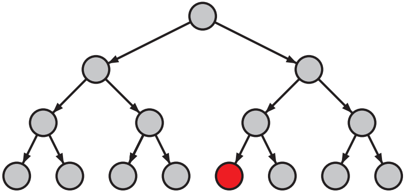


Semantics of eventually and always

$\forall\Diamond\Phi$

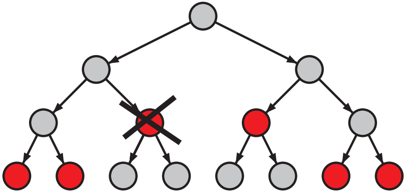


$\exists\Diamond\Phi$

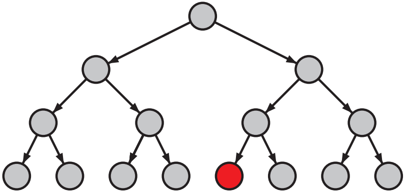


Semantics of eventually and always

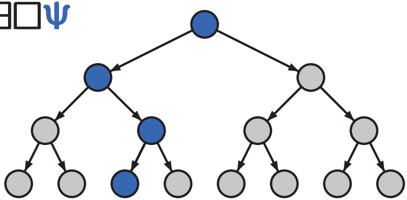
$\neg\forall\Diamond\Phi$



$\exists\Diamond\Phi$

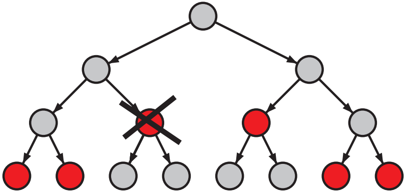


$\exists\Box\Psi$

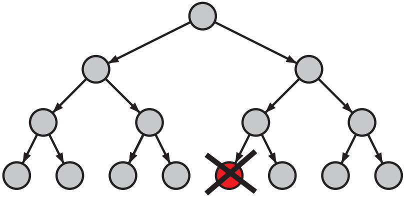


Semantics of eventually and always

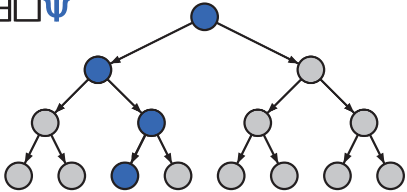
$\neg\forall\Diamond\Phi$



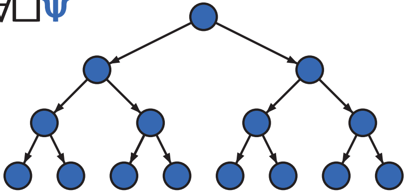
$\neg\exists\Diamond\Phi$



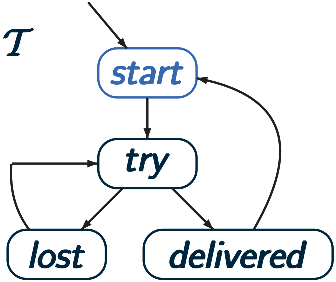
$\exists\Box\Psi$



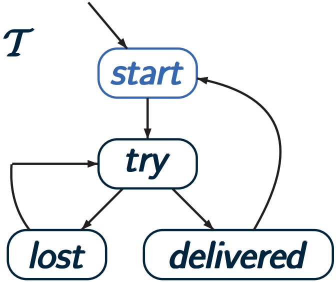
$\forall\Box\Psi$



Example for CTL semantics



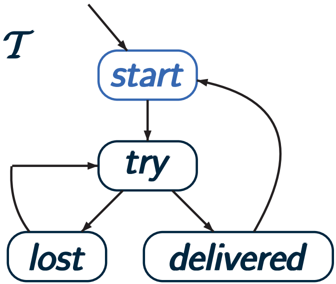
Example for CTL semantics



CTL formula

$\Phi = \forall\Box\forall\Diamond start$

Example for CTL semantics

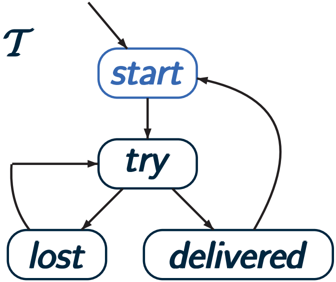


CTL formula

$\Phi = \forall\Box \; \forall\Diamond start$

$Sat(\forall\Diamond start) = ?$

Example for CTL semantics

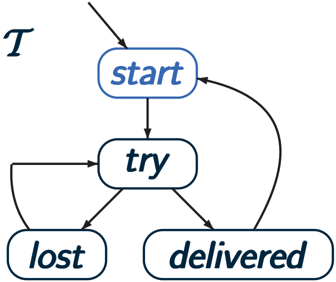


CTL formula

$\Phi = \forall\Box \; \forall\Diamond start$

$$Sat(\forall\Diamond start) = \{start, delivered\}$$

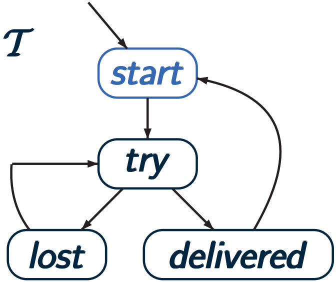
Example for CTL semantics



CTL formula

$\Phi = \forall\Box\forall\Diamond start \;\hat{=}\; \forall\Box(start \vee delivered)$

$Sat(\forall\Diamond start) = \{start, delivered\}$



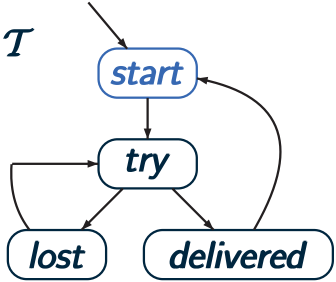
CTL formula

$\Phi = \forall\Box\forall\Diamond start \quad \widehat{=} \quad \forall\Box(start \vee delivered)$

$$Sat(\forall\Diamond start) = \{start, delivered\}$$

$$Sat(\Phi) = \varnothing$$

Example for CTL semantics



$\mathcal{T} \not\models \forall\Box\forall\Diamond start$

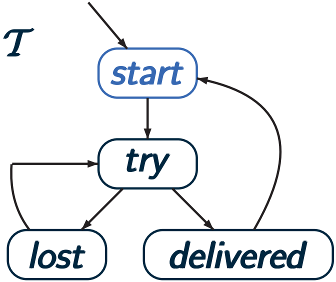
CTL formula

$\Phi = \forall\Box\forall\Diamond start \quad \widehat{=} \quad \forall\Box(start \vee delivered)$

$$Sat(\forall\Diamond start) = \{start, delivered\}$$

$$Sat(\Phi) = \varnothing$$

Example for CTL semantics



$\mathcal{T} \not\models \forall\Box\forall\Diamond start$

"infinitely often *start*"

CTL formula

$\Phi = \forall\Box\forall\Diamond start \;\; \widehat{=} \;\; \forall\Box(start \vee delivered)$

$Sat(\forall\Diamond start) = \{start, delivered\}$

$Sat(\Phi) = \varnothing$

Specifying "infinitely often" in CTL

Specifying "infinitely often" in CTL

If s is a state in a TS and $a \in AP$ then:

$$s \models_{\mathrm{CTL}} \forall\Box\forall\Diamond a$$

iff for all paths $\pi = s_0\, s_1\, s_2 \ldots \in Paths(s)$:

$$\overset{\infty}{\exists}\, i \geq 0. \quad \text{s.t.} \quad s_i \models a$$

If s is a state in a TS and $a \in AP$ then:

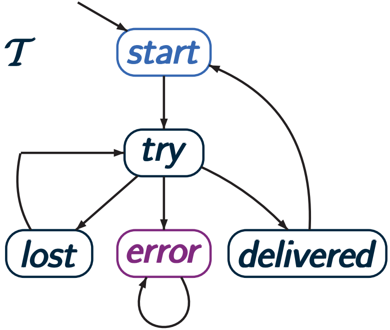
$$s \models_{\text{CTL}} \forall\Box\forall\Diamond a$$

iff for all paths $\pi = s_0\, s_1\, s_2 \ldots \in Paths(s)$:

$$\overset{\infty}{\exists}\, i \geq 0. \quad \text{s.t.} \quad s_i \models a$$

iff $s \models_{\text{LTL}} \Box\Diamond a$

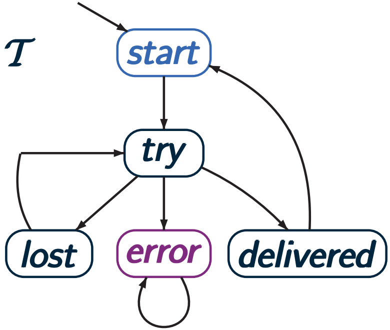
Example: CTL semantics



$\Phi_1 = \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

Example: CTL semantics

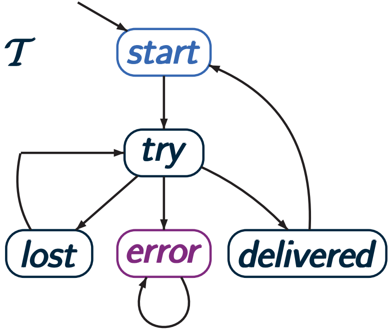


$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



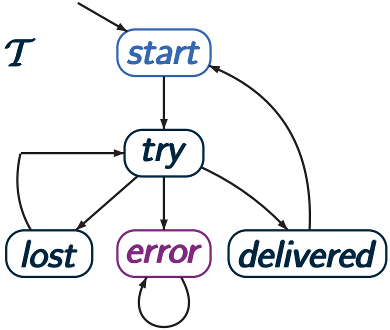
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$?

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) =$?

Example: CTL semantics



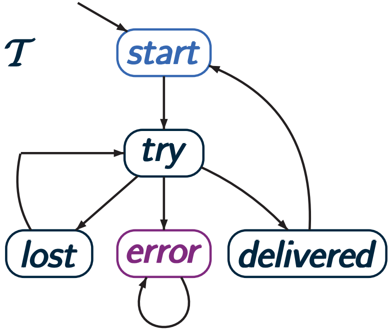
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$ ✓

$\Phi_1 = \exists\Diamond \boxed{\forall\Box\neg start} \rightsquigarrow \exists\Diamond \boxed{error}$

$Sat(\forall\Box\neg start) = \{error\}$

$Sat(\exists\Diamond\forall\Box\neg start) = Sat(\exists\Diamond error) =$ "all states"

Example: CTL semantics

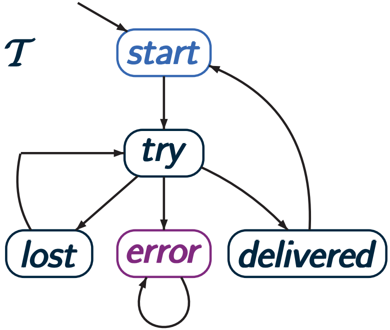


$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**

Example: CTL semantics



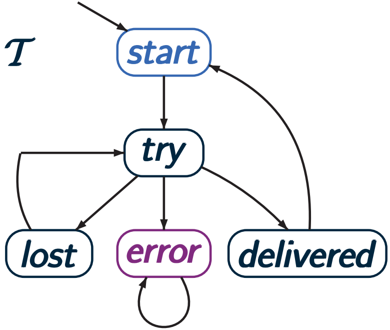
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**

$\Phi_2 = \forall\bigcirc\exists\bigcirc\ \boxed{\forall\Box\neg start}$

$$Sat(\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

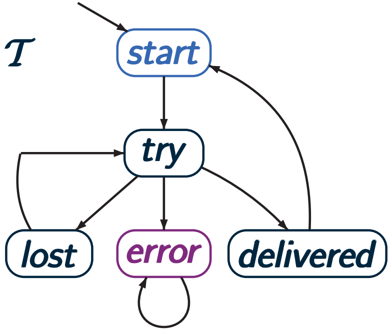
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ **?**

$\Phi_2 = \forall\bigcirc\ \boxed{\exists\bigcirc\forall\Box\neg start} \quad \rightsquigarrow \quad \forall\bigcirc\ \boxed{\exists\bigcirc error}$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$

Example: CTL semantics

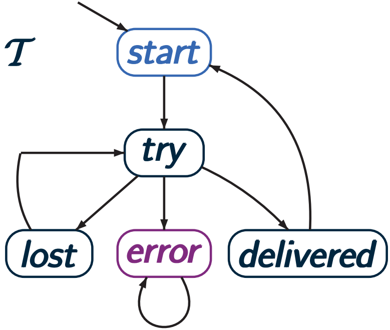


$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$?

$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start \quad \rightsquigarrow \quad \forall\bigcirc(error \vee try)$

$$Sat(\forall\Box\neg start) = \{error\}$$
$$Sat(\exists\bigcirc\forall\Box\neg start) = \{error, try\}$$
$$Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) = \ ?$$



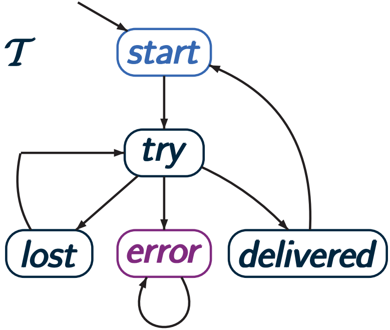
$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$ ✓

$\Phi_2 = \forall\bigcirc\exists\bigcirc\forall\Box\neg start \quad \rightsquigarrow \quad \boxed{\forall\bigcirc(error \vee try)}$

$$
\begin{aligned}
Sat(\forall\Box\neg start) &= \{error\} \\
Sat(\exists\bigcirc\forall\Box\neg start) &= \{error, try\} \\
Sat(\forall\bigcirc\exists\bigcirc\forall\Box\neg start) &= \{error, lost, start\}
\end{aligned}
$$

Example: CTL semantics



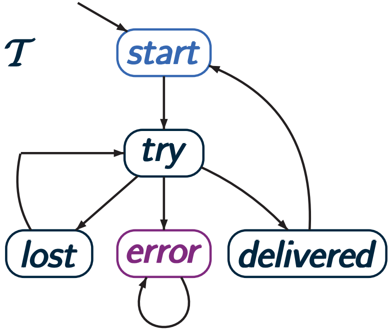
$$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start$$

$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

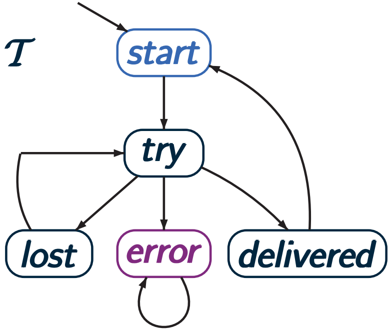
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$?

$\Phi_3 = \exists\bigcirc\forall\bigcirc \boxed{\forall\Box\neg start}$

$Sat(\forall\Box\neg start) = \{error\}$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

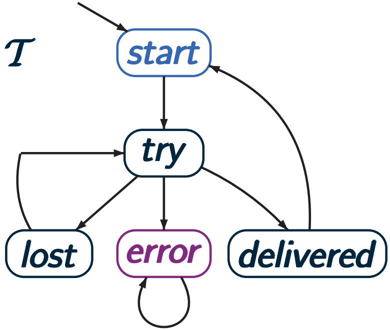
$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

$$\Phi_3 = \exists\bigcirc \boxed{\forall\bigcirc\forall\Box\neg start} \rightsquigarrow \exists\bigcirc \boxed{\forall\bigcirc error}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\forall\bigcirc\forall\Box\neg start) = \{error\}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

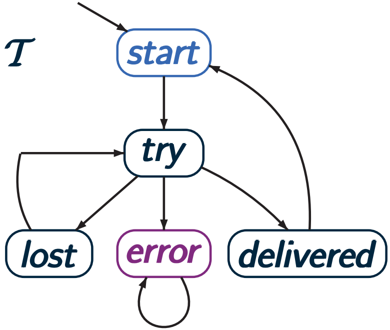
$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$ **?**

$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start \rightsquigarrow$ $\exists\bigcirc error$

$$\begin{aligned} Sat(\forall\Box\neg start) &= \{error\} \\ Sat(\forall\bigcirc\forall\Box\neg start) &= \{error\} \\ Sat(\exists\bigcirc\forall\bigcirc\forall\Box\neg start) &= \ ? \end{aligned}$$

Example: CTL semantics



$\mathcal{T} \models \exists\Diamond\forall\Box\neg start$

$\mathcal{T} \models \forall\bigcirc\exists\bigcirc\forall\Box\neg start$

$\mathcal{T} \not\models \exists\bigcirc\forall\bigcirc\forall\Box\neg start$

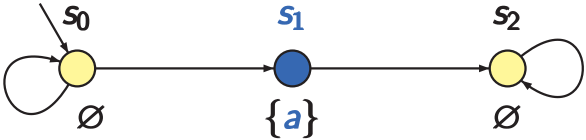
$$\Phi_3 = \exists\bigcirc\forall\bigcirc\forall\Box\neg start \rightsquigarrow \boxed{\exists\bigcirc error}$$

$$Sat(\forall\Box\neg start) = \{error\}$$

$$Sat(\forall\bigcirc\forall\Box\neg start) = \{error\}$$

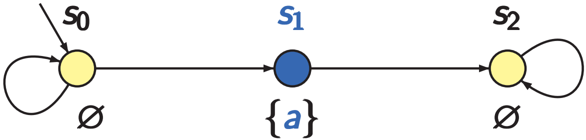
$$Sat(\exists\bigcirc\forall\bigcirc\forall\Box\neg start) = \{error, try\}$$

Example: CTL semantics



does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

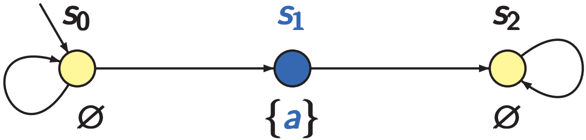
Example: CTL semantics



does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

Example: CTL semantics

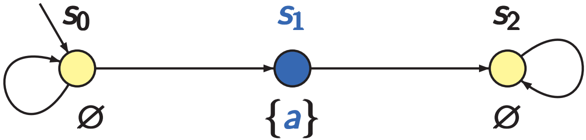


does $\mathcal{T} \models \exists \bigcirc \forall \Box \neg a$ hold ?

answer: no

$$Sat(\forall \Box \neg a) \;=\; \{s_2\}$$

Example: CTL semantics

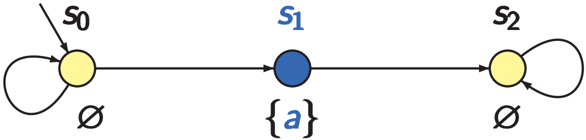


does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

$$Sat(\forall\Box\neg a) = \{s_2\}$$
$$Sat(\exists\bigcirc\forall\Box\neg a) = \{s_2, s_1\}$$

Example: CTL semantics

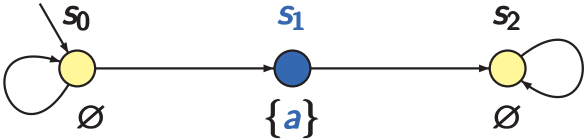


does $\mathcal{T} \models \exists \bigcirc \forall \Box \neg a$ hold ?

answer: no

does $\mathcal{T} \models \forall \Box \exists \bigcirc \neg a$ hold ?

Example: CTL semantics



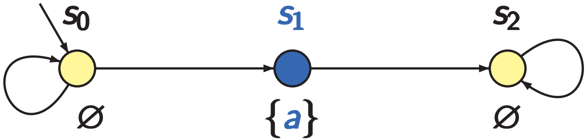
does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

does $\mathcal{T} \models \forall\Box\exists\bigcirc\neg a$ hold ?

answer: yes

Example: CTL semantics



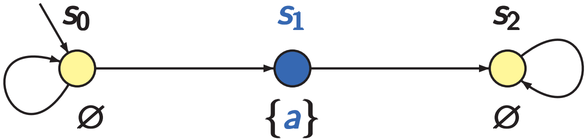
does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

answer: no

does $\mathcal{T} \models \forall\Box\exists\bigcirc\neg a$ hold ?

answer: yes

$$Sat(\exists\bigcirc\neg a) \;=\; \{s_0, s_1, s_2\}$$



does $\mathcal{T} \models \exists\bigcirc\forall\Box\neg a$ hold ?

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does $\mathcal{T} \models \forall\Box\exists\bigcirc\neg a$ hold ?

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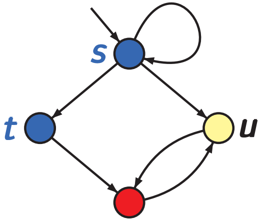
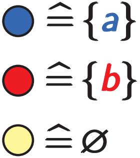
$$\begin{aligned} Sat(\exists\bigcirc\neg a) &= \{s_0, s_1, s_2\} \\ Sat(\forall\Box\exists\bigcirc\neg a) &= \{s_0, s_1, s_2\} \end{aligned}$$

Example: CTL semantics





$\mathcal{T} \models \exists\Box\exists(a \,\mathsf{U}\, b)$?

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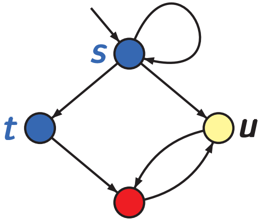

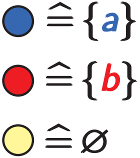

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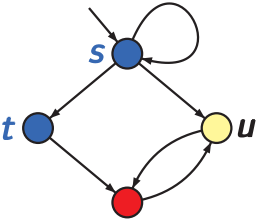

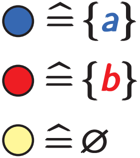

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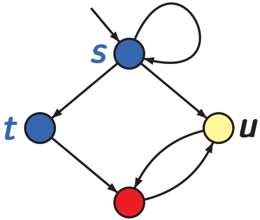

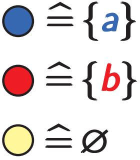

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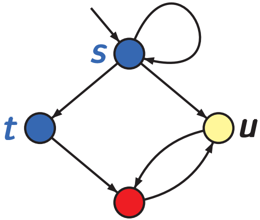

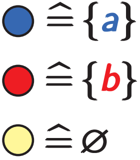

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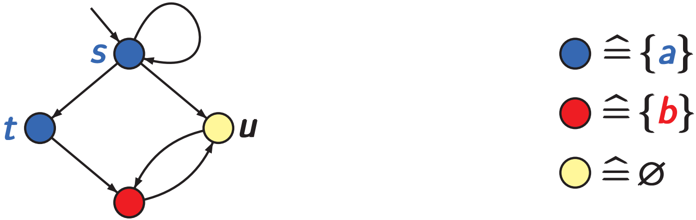

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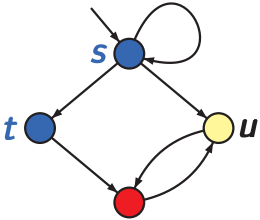

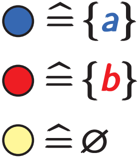

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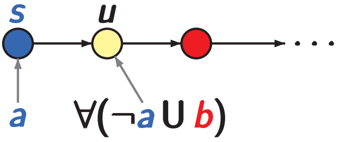

$\models a \,\mathsf{U}\, \forall(\neg a \,\mathsf{U}\, b)$

Correct or wrong?

Let $\mathcal{T}$ be a transition system and Φ a CTL formula.
Is the following statement correct **?**

$$\text{if } \mathcal{T} \not\models \neg\Phi \text{ then } \mathcal{T} \models \Phi$$

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answer: no

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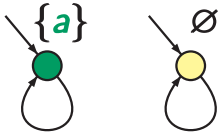
transition system $\mathcal{T}$ with 2 initial states:



$\mathcal{T} \not\models \exists\Box a$

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Hamilton path problem

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question: does G have a Hamilton path, i.e., a path that visits each node exactly once **?**

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CTL-encoding of the Hamilton path problem

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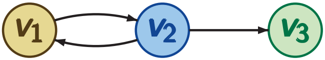

digraph G

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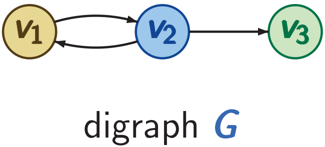
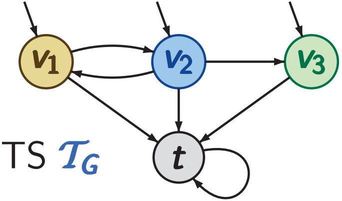


$\rightsquigarrow$



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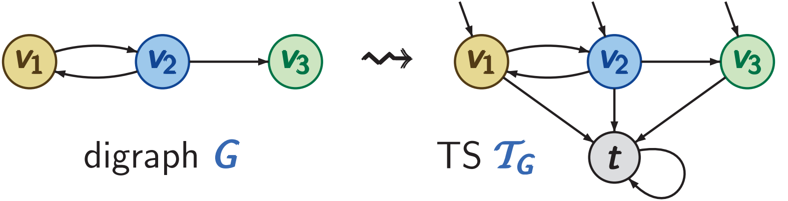

CTL formula Φ

$$(v_1 \wedge \exists\bigcirc(v_2 \wedge \exists\bigcirc v_3)) \vee (v_1 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_2)) \vee$$
$$(v_2 \wedge \exists\bigcirc(v_1 \wedge \exists\bigcirc v_3)) \vee (v_2 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_1)) \vee$$
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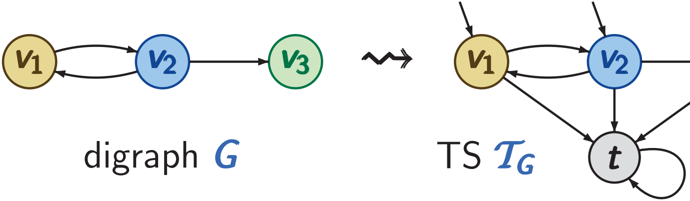
CTL-encoding of the Hamilton path problem

finite digraph $\mathcal{G}$	$\leadsto$	finite TS $\mathcal{T}_{\mathcal{G}}$ + CTL formula Φ

s.t. $\mathcal{G}$ has a Hamilton path iff $\mathcal{T}_{\mathcal{G}} \not\models \Phi$



digraph $\mathcal{G}$ TS $\mathcal{T}_{\mathcal{G}}$

CTL formula Φ = negation of the formula

$$(v_1 \wedge \exists\bigcirc(v_2 \wedge \exists\bigcirc v_3)) \vee (v_1 \wedge \exists\bigcirc(v_3 \wedge \exists\bigcirc v_2)) \vee$$
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Examples:

$$\neg\neg\Phi \equiv \Phi$$

$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$

Equivalence of CTL formulas

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$$\vdots$$

$$\neg\forall\bigcirc\Phi \equiv \exists\bigcirc\neg\Phi$$

Correct or wrong?

$$\exists\Diamond(a \wedge b) \equiv \exists\Diamond a \wedge \exists\Diamond b$$

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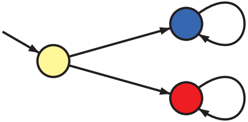

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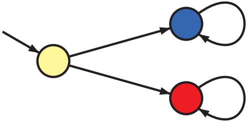

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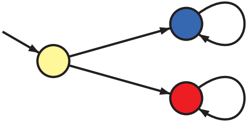

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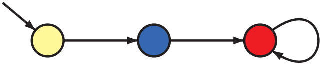

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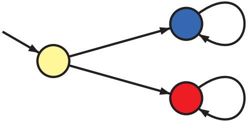

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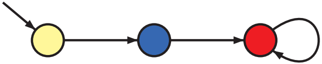

but:

$$\forall\Box(\Phi_1 \wedge \Phi_2) \equiv \forall\Box\Phi_1 \wedge \forall\Box\Phi_2$$

$$\exists\Diamond(\Phi_1 \vee \Phi_2) \equiv \exists\Diamond\Phi_1 \vee \exists\Diamond\Phi_2$$

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$$\forall \bigcirc \forall \Box a \equiv \forall \Box \forall \bigcirc a$$

Correct or wrong?

$\forall\bigcirc\forall\Box a \equiv \forall\Box\forall\bigcirc a$

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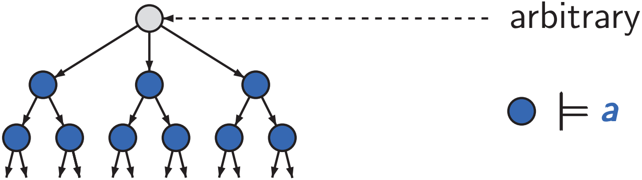
both formulas require computation trees of the form:



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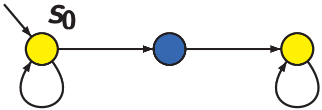

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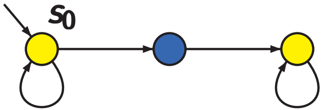

$s_0 \not\models \exists\bigcirc\exists\Box a$

Correct or wrong?

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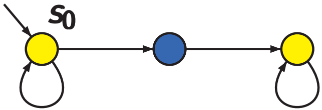

$s_0 \not\models \exists\bigcirc\exists\Box a$

note: $Sat(\exists\Box a) = \varnothing$

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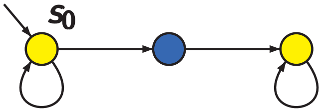

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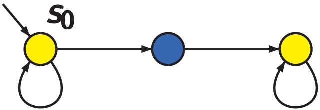

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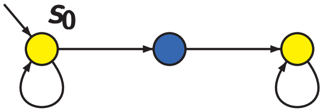

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$s_0 \models \exists \bigcirc a$

$\Longrightarrow \quad s_0\, s_0\, s_0 \ldots \models \Box \exists \bigcirc a$

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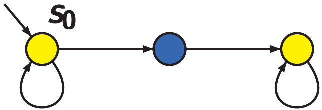
correct.

$$\exists\bigcirc\exists\Box a \;\equiv\; \exists\Box\exists\bigcirc a$$

wrong, e.g.,



$$s_0 \not\models \exists\bigcirc\exists\Box a$$
$$s_0 \models \exists\Box\exists\bigcirc a$$

$$s_0 \models \exists\bigcirc a$$
$$\Longrightarrow \quad s_0\, s_0\, s_0 \ldots \models \Box\exists\bigcirc a$$
$$\Longrightarrow \quad s_0 \models \exists\Box\exists\bigcirc a$$

Weak until W

Weak until W

in **LTL**: $\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

in **CTL**: **?**

in **LTL**: $\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \lor \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \land \neg\psi) \, \mathsf{W} \, (\neg\varphi \land \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \land \neg\psi) \, \mathsf{U} \, (\neg\varphi \land \neg\psi)$$

in **CTL**: **?**

in **LTL**: $\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\forall((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

in **LTL**: $\varphi \, \mathsf{W} \, \psi \stackrel{\text{def}}{=} (\varphi \, \mathsf{U} \, \psi) \vee \Box\varphi$

duality of **U** and **W**:

$$\neg(\varphi \, \mathsf{U} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{W} \, (\neg\varphi \wedge \neg\psi)$$

$$\neg(\varphi \, \mathsf{W} \, \psi) \equiv (\varphi \wedge \neg\psi) \, \mathsf{U} \, (\neg\varphi \wedge \neg\psi)$$

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\forall((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\exists((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

Weak until W in CTL

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\forall\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \overset{\text{def}}{=} \neg\exists\big((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)\big)$$

Weak until W in CTL

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, W \, \Psi) \stackrel{\text{def}}{=} \neg\forall(\,(\Phi \wedge \neg\Psi) \, U \, (\neg\Phi \wedge \neg\Psi)\,)$$

$$\forall(\Phi \, W \, \Psi) \stackrel{\text{def}}{=} \neg\exists(\,(\Phi \wedge \neg\Psi) \, U \, (\neg\Phi \wedge \neg\Psi)\,)$$

note that:

$$\exists(\Phi \, W \, \Psi) \equiv \exists(\Phi \, U \, \Psi) \vee \exists\Box\Phi$$

Weak until W in CTL

definition of **W** in **CTL** on the basis of duality rules:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\forall((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

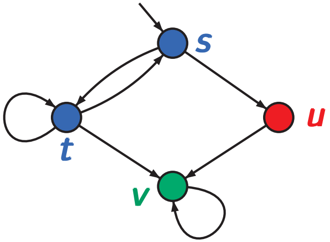
$$\forall(\Phi \, \mathsf{W} \, \Psi) \stackrel{\text{def}}{=} \neg\exists((\Phi \wedge \neg\Psi) \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi))$$

note that:

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \exists(\Phi \, \mathsf{U} \, \Psi) \vee \exists\Box\Phi$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \not\equiv \forall(\Phi \, \mathsf{U} \, \Psi) \vee \forall\Box\Phi$$

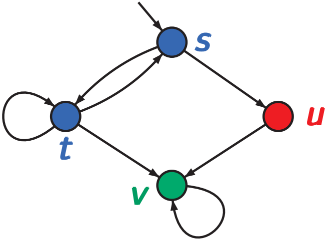
Weak until W in CTL

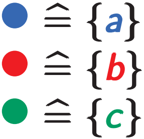


$\bullet \;\widehat{=}\; \{a\}$
$\bullet \;\widehat{=}\; \{b\}$
$\bullet \;\widehat{=}\; \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ **?**

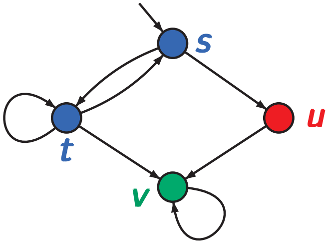
Weak until W in CTL





$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ $\checkmark$ as $s \models \exists(a \,\mathsf{W}\, c)$

Weak until W in CTL



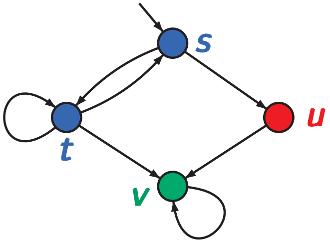
$\bullet \mathrel{\widehat{=}} \{a\}$

$\bullet \mathrel{\widehat{=}} \{b\}$

$\bullet \mathrel{\widehat{=}} \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ $\surd$ as $s\, s_1\, s_2 \ldots \models \Diamond\exists(a \mathsf{W} c)$

Weak until W in CTL



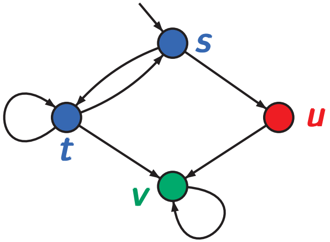
● $\widehat{=}$ $\{a\}$

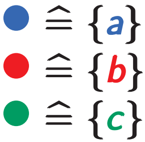
● $\widehat{=}$ $\{b\}$

● $\widehat{=}$ $\{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ $\surd$ as $s \models \exists(a \mathsf{W} c)$

$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$?

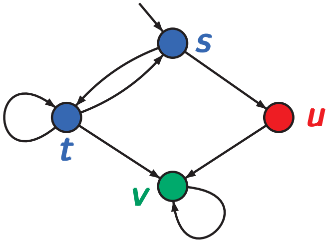




$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ ✓ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ ✓ as $s \models \exists\Box a$

Weak until W in CTL



$\bullet \,\widehat{=}\, \{a\}$

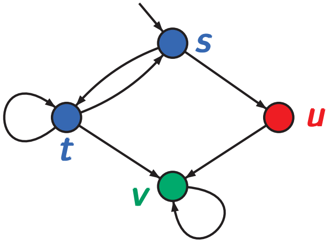
$\bullet \,\widehat{=}\, \{b\}$

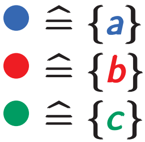
$\bullet \,\widehat{=}\, \{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ $\surd$ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ $\surd$ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$?



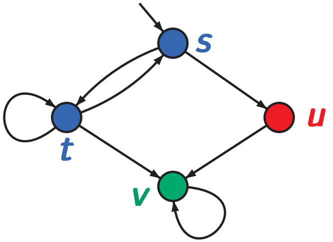


$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ √ as $s \models \exists(a \mathsf{W} c)$

$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ √ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathsf{W} (a \wedge b))$ √

Weak until W in CTL



● $\widehat{=}$ $\{a\}$

● $\widehat{=}$ $\{b\}$

● $\widehat{=}$ $\{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ ✓ as $s \models \exists(a \mathsf{W} c)$

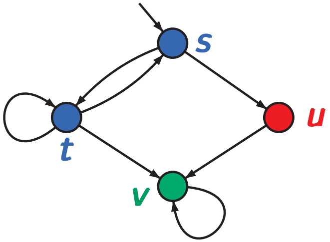
$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ ✓ as $s \models \exists\Box a$

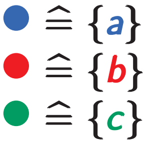
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathsf{W} (a \wedge b))$ ✓

$\uparrow$

three types of paths: $(s\,t)^\omega$ or $(s\,t)^+ v^\omega$ or $(s\,t)^* s\,u\,v^\omega$

Weak until W in CTL





$\mathcal{T} \models \forall\Diamond\exists(a \mathsf{W} c)$ ✓ as $s \models \exists(a \mathsf{W} c)$

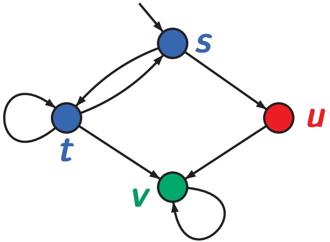
$\mathcal{T} \models \exists(a \mathsf{W} \exists\Box b)$ ✓ as $s \models \exists\Box a$

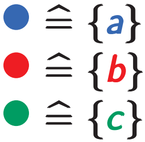
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \mathsf{W} (a \wedge b))$ ✓

three types of paths: $(s\,t)^\omega$ or $(s\,t)^+ v^\omega$ or $(s\,t)^* s\,u\,v^\omega$

in all three cases: $\pi \models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

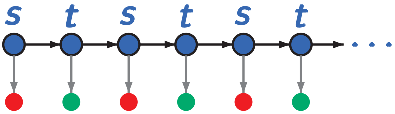




$\mathcal{T} \models \forall\Diamond\exists(a \, \mathsf{W} \, c)$ $\surd$ as $s \models \exists(a \, \mathsf{W} \, c)$

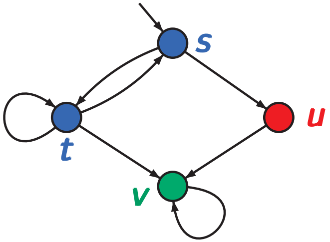
$\mathcal{T} \models \exists(a \, \mathsf{W} \, \exists\Box b)$ $\surd$ as $s \models \exists\Box a$

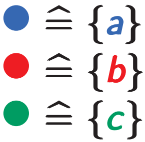
$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \, \mathsf{W} \, (a \wedge b))$ $\surd$



$\models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

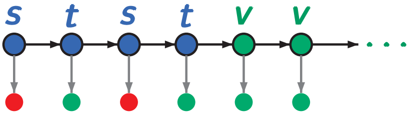




$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ ✓ as $s \models \exists(a \,\mathsf{W}\, c)$

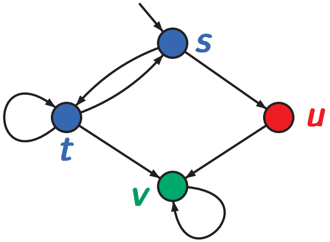
$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ ✓ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$ ✓



$\models \Box\exists\bigcirc(b \vee c)$

Weak until W in CTL

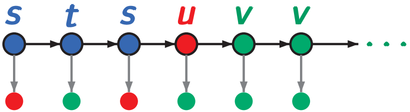


● $\widehat{=}$ $\{a\}$
● $\widehat{=}$ $\{b\}$
● $\widehat{=}$ $\{c\}$

$\mathcal{T} \models \forall\Diamond\exists(a \,\mathsf{W}\, c)$ √ as $s \models \exists(a \,\mathsf{W}\, c)$

$\mathcal{T} \models \exists(a \,\mathsf{W}\, \exists\Box b)$ √ as $s \models \exists\Box a$

$\mathcal{T} \models \forall((\exists\bigcirc(b \vee c)) \,\mathsf{W}\, (a \wedge b))$ √



$\models \Box\exists\bigcirc(b \vee c)$

Expansion laws

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists \bigcirc \exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv ?$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \;\equiv\; \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \;\equiv\; \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \;\equiv\; \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \,\mathsf{W}\, \Psi) \;\equiv$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{W}\, \Psi))$$

$$\forall(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{W}\, \Psi))$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{W} \, \Psi))$$

$$\exists\Box\Phi \equiv ?$$

Expansion laws

$$\exists(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{U} \, \Psi))$$

$$\forall(\Phi \, \mathsf{U} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{U} \, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \, \mathsf{W} \, \Psi))$$

$$\forall(\Phi \, \mathsf{W} \, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \, \mathsf{W} \, \Psi))$$

$$\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$$

Expansion laws

$$\exists(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi))$$

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{U}\, \Psi))$$

$$\exists\Diamond\Psi \equiv \Psi \vee \exists\bigcirc\exists\Diamond\Psi$$

$$\forall\Diamond\Psi \equiv \Psi \vee \forall\bigcirc\forall\Diamond\Psi$$

$$\exists(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \exists\bigcirc\exists(\Phi \,\mathsf{W}\, \Psi))$$

$$\forall(\Phi \,\mathsf{W}\, \Psi) \equiv \Psi \vee (\Phi \wedge \forall\bigcirc\forall(\Phi \,\mathsf{W}\, \Psi))$$

$$\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$$

$$\forall\Box\Phi \equiv \Phi \wedge \forall\bigcirc\forall\Box\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

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duality of **U** and **W**, e.g.:

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

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self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi\, \mathbf{U}\, \Psi)$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi \, \mathbf{U} \, \Psi) \equiv \neg\exists((\Phi \wedge \neg\Psi) \, \mathbf{W} \, (\neg\Phi \wedge \neg\Psi))$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

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$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W**, e.g.:

$$\forall(\Phi\,\mathbf{U}\,\Psi) \equiv \neg\exists((\Phi \wedge \neg\Psi)\,\mathbf{W}\,(\neg\Phi \wedge \neg\Psi))$$

$$\equiv \neg\exists((\neg\Psi)\,\mathbf{W}\,(\neg\Phi \wedge \neg\Psi))$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall \Box \Phi \equiv \neg \exists \Diamond \neg \Phi$$

$$\forall \Diamond \Phi \equiv \neg \exists \Box \neg \Phi$$

self-duality of $\bigcirc$:

$$\forall \bigcirc \Phi \equiv \neg \exists \bigcirc \neg \Phi$$

$$\exists \bigcirc \Phi \equiv \neg \forall \bigcirc \neg \Phi$$

duality of **U** and **W**, e.g.:

$$\begin{aligned} \forall(\Phi \, \mathsf{U} \, \Psi) &\equiv \neg \exists((\Phi \wedge \neg \Psi) \, \mathsf{W} \, (\neg \Phi \wedge \neg \Psi)) \\ &\equiv \neg \exists((\neg \Psi) \, \mathsf{W} \, (\neg \Phi \wedge \neg \Psi)) \\ &\equiv \neg \exists((\neg \Psi) \, \mathsf{U} (\neg \Phi \wedge \neg \Psi)) \;\wedge\; \neg \exists \Box \neg \Psi \end{aligned}$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

duality of **U** and **W** yields

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \neg\exists((\neg\Psi)\,\mathsf{U}(\neg\Phi \wedge \neg\Psi)) \wedge \neg\exists\Box\neg\Psi$$

Duality laws

duality of $\Box$ and $\Diamond$:

$$\forall\Box\Phi \equiv \neg\exists\Diamond\neg\Phi$$

$$\forall\Diamond\Phi \equiv \neg\exists\Box\neg\Phi$$

self-duality of $\bigcirc$:

$$\forall\bigcirc\Phi \equiv \neg\exists\bigcirc\neg\Phi$$

$$\exists\bigcirc\Phi \equiv \neg\forall\bigcirc\neg\Phi$$

derivation of $\forall \mathsf{U}$ from $\exists \mathsf{U}$ and $\exists\Box$:

$$\forall(\Phi \,\mathsf{U}\, \Psi) \equiv \neg\exists((\neg\Psi)\,\mathsf{U}(\neg\Phi \wedge \neg\Psi)) \wedge \neg\exists\Box\neg\Psi$$

$\forall \mathsf{U}$ and $\forall\bigcirc$ are expressible via $\exists \mathsf{U}$, $\exists\bigcirc$ and $\exists\Box$