Decision Procedures
An Algorithmic Point of View

Equalities and Uninterpreted Functions

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Part III

Equalities and Uninterpreted Functions

Outline

- A Boolean combination of Equalities and Propositions

$$x_1 = x_2 \land (x_2 = x_3 \lor \neg((x_1 = x_3) \land b \land x_1 = 2))$$

- We always push negations inside (NNF):

$$x_1 = x_2 \land (x_2 = x_3 \lor ((x_1 \neq x_3) \land \neg b \land x_1 \neq 2))$$

Syntax of Equality Logic

$$
\begin{aligned}
formula \quad : & \quad formula \lor formula \\
& \mid \quad \neg formula \\
& \mid \quad atom \\
\\
atom \quad : & \quad \textit{term-variable} = \textit{term-variable} \\
& \mid \quad \textit{term-variable} = constant \\
& \mid \quad \textit{Boolean-variable}
\end{aligned}
$$

- The *term-variables* are defined over some (possible infinite) domain. The constants are from the same domain.
- The set of Boolean variables is always separate from the set of term variables

- Allows more natural description of systems, although technically it is as expressible as Propositional Logic.
- Obviously NP-hard.
- In fact, it is in NP, and hence NP-complete, for reasons we shall see later.

$$
\begin{array}{rcl}
\textit{formula} & : & \textit{formula} \lor \textit{formula} \\
& | & \neg \textit{formula} \\
& | & \textit{atom} \\
\\
\textit{atom} & : & \textit{term} = \textit{term} \\
& | & \textit{Boolean-variable} \\
\\
\textit{term} & : & \textit{term-variable} \\
& | & \textit{function} \, (\, \text{list of } \textit{term}\text{s} \,)
\end{array}
$$

The *term-variables* are defined over some (possible infinite) domain.
Constants are functions with an empty list of terms.

- Every function is a mapping from a domain to a range.
- Example: the '+' function over the naturals $\mathbb{N}$ is a mapping from $\langle \mathbb{N} \times \mathbb{N} \rangle$ to $\mathbb{N}$.

- Suppose we replace '+' by an uninterpreted binary function $f(a, b)$
- Example:

$$x_1 + x_2 = x_3 + x_4 \quad \text{is replaced by} \quad f(x_1, x_2) = f(x_3, x_4)$$

- We lost the 'semantics' of '+', as f can represent any binary function.

- 'Loosing the semantics' means that f is not restricted by any axioms or rules of inference.
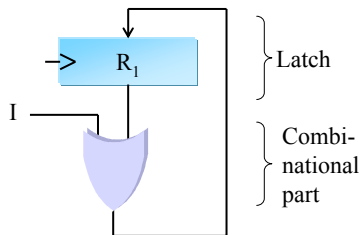- But f is still a function!

- The most general axiom for any function is functional consistency.
- Example: if $x = y$, then $f(x) = f(y)$ for any function f.

- Functional consistency axiom schema:

$$x_1 = x'_1 \wedge \ldots \wedge x_n = x'_n \quad \implies \quad f(x_1, \ldots, x_n) = f(x'_1, \ldots, x'_n)$$

- Sometimes, functional consistency is all that is needed for a proof.

- Circuits consist of combinational gates and latches (registers)
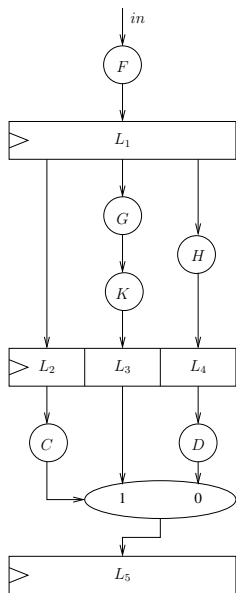
Example: Circuit Transformations

- Circuits consist of combinational gates and latches (registers)

- The combinational gates can be modeled using functions
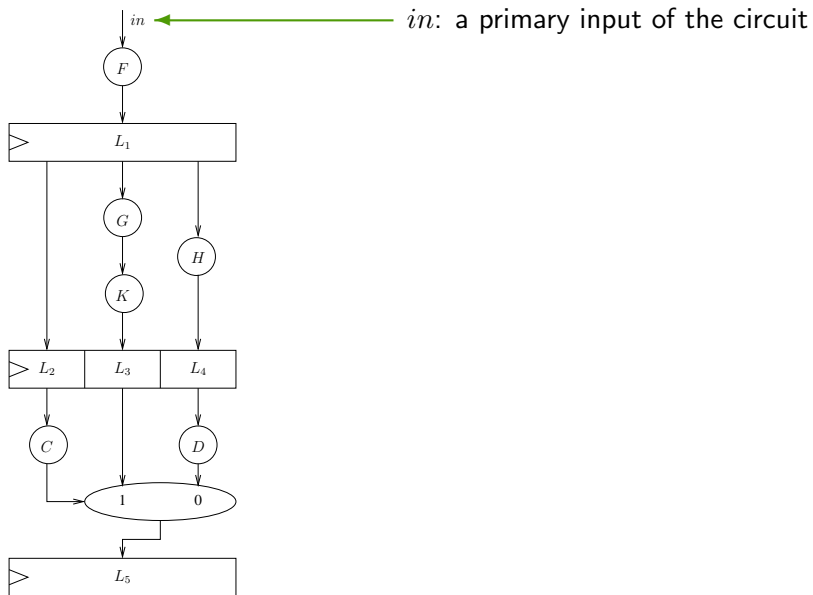- The latches can be modeled with variables
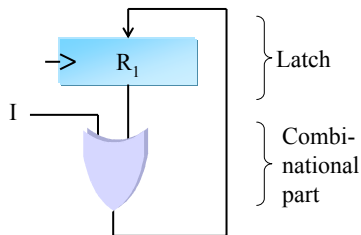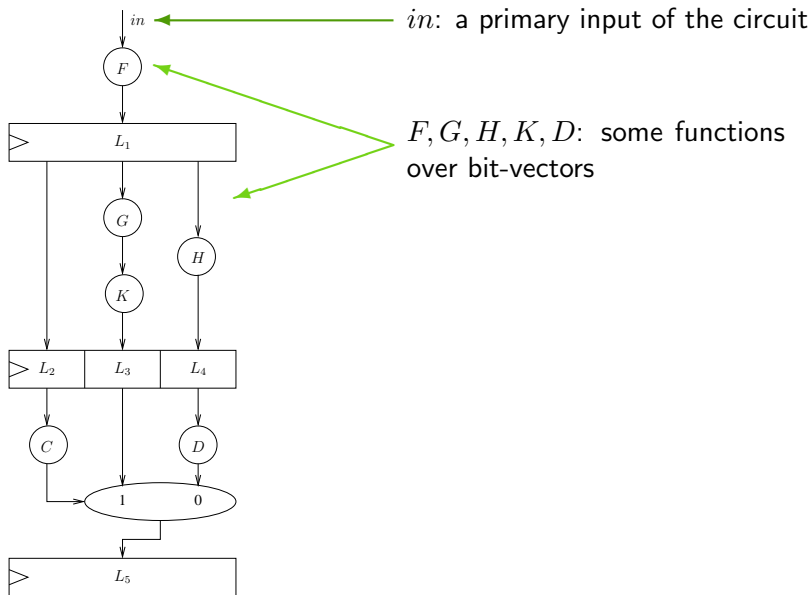
$$f(x, y) \quad := \quad x \vee y$$
$$R_1' \quad = \quad f(R_1, I)$$

Example: Circuit Transformations

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in: a primary input of the circuit

in: a primary input of the circuit

F, G, H, K, D: some functions over bit-vectors

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$L_1, \ldots, L_5$: latches (registers)

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$L_1, \ldots, L_5$: latches (registers)

C: a predicate over bit-vectors

a multiplexer (case-split)

Example: Circuit Transformations

- A pipeline processes data in *stages*
- Data is processed in parallel – as in an assembly line
- Formal model:

$$
\begin{aligned}
L_1 &= f(I) \\
L_2 &= L_1 \\
L_3 &= k(g(L_1)) \\
L_4 &= h(L_1) \\
L_5 &= c(L_2)\,?\,L_3\,:\,l(L_4)
\end{aligned}
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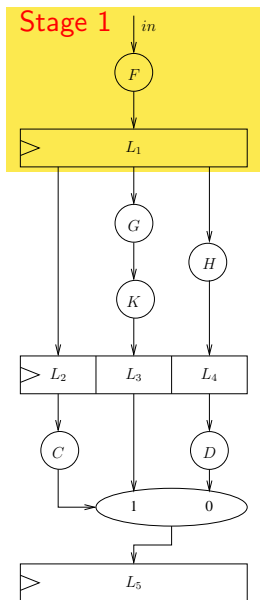- Data is processed in parallel – as in an assembly line
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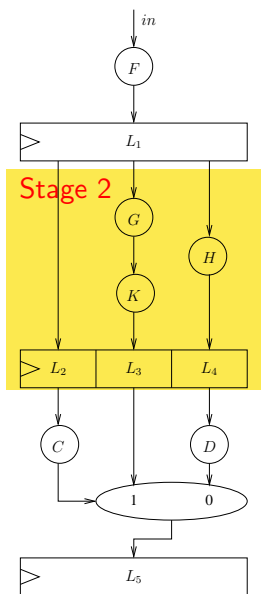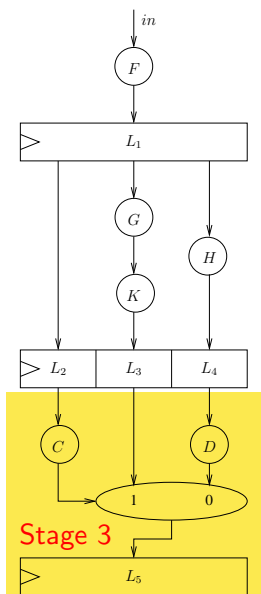- Data is processed in parallel – as in an assembly line
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- The maximum clock frequency depends on the longest path between two latches
- Note that the output of g is used as input to k
- We want to speed up the design by postponing k to the third stage
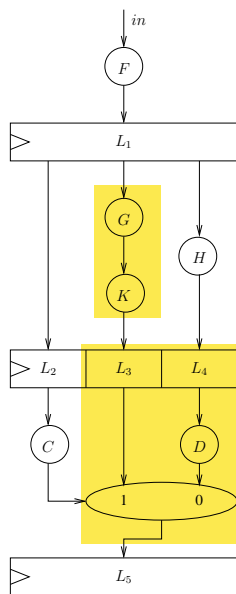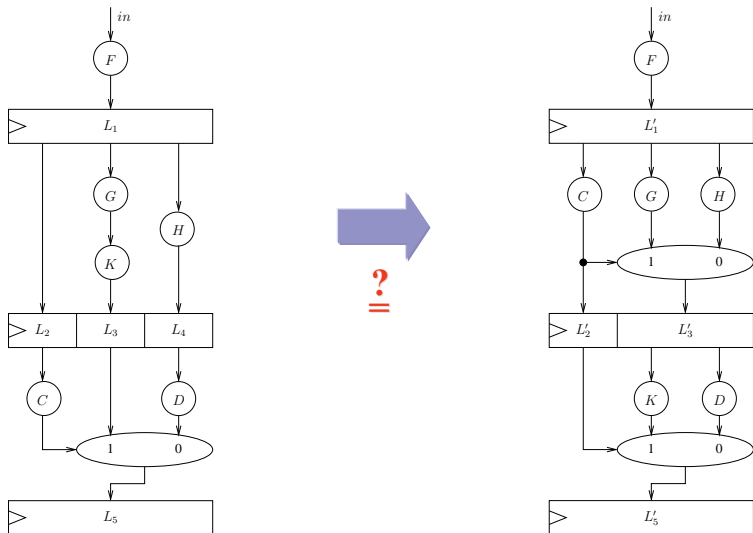
- The maximum clock frequency depends on the longest path between two latches
- Note that the output of g is used as input to k
- We want to speed up the design by postponing k to the third stage
- Also note that the circuit only uses one of L_3 or L_4, never both
- ⇒ We can remove one of the latches

Example: Circuit Transformations

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$$

$$
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L_1' &= f(I) \\
L_2' &= c(L_1') \\
L_3' &= c(L_1') \, ? \, g(L_1') \, : \, h(L_1') \\
L_5' &= L_2' \, ? \, k(L_3') \, : \, l(L_3')
\end{aligned}
$$

$$
L_5 \overset{?}{=} L_5'
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\end{aligned}
$$

$$
L_5 \stackrel{?}{=} L_5'
$$

- Equivalence in this case holds regardless of the actual functions

- Conclusion: can be decided using *Equality Logic and Uninterpreted Functions*

- Given: a formula φ^{UF} with uninterpreted functions
- For each function in φ^{UF}:

  1. Number function instances $\longrightarrow F_2(\,F_1(x)\,) = 0$
     (from the inside out)

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2. Replace each function instance with a new variable $\longrightarrow$ $f_2 = 0$

3. Add functional consistency constraint to φ^{UF} for every pair of instances of the same function. $\longrightarrow$ $((x = f_1) \longrightarrow (f_2 = f_1))$ $\longrightarrow f_2 = 0$

Ackermann's reduction: Example

Suppose we want to check

$$x_1 \neq x_2 \vee F(x_1) = F(x_2) \vee F(x_1) \neq F(x_3)$$

for validity.

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2. Replace each function with a new variable:

$$x_1 \neq x_2 \vee f_1 = f_2 \vee f_1 \neq f_3$$

3. Add functional consistency constraints:

$$\left(\begin{array}{ll} (x_1 = x_2 \rightarrow f_1 = f_2) & \wedge \\ (x_1 = x_3 \rightarrow f_1 = f_3) & \wedge \\ (x_2 = x_3 \rightarrow f_2 = f_3) \end{array} \right) \rightarrow$$

$$((x_1 \neq x_2) \vee (f_1 = f_2) \vee (f_1 \neq f_3))$$

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      F_i with an expression F_i^*

$$F_i^* := \begin{pmatrix} \text{case} & x_1 = x_i & : f_1 \\ & x_2 = x_i & : f_2 \\ & \vdots & \\ & x_{i-1} = x_i : f_{i-1} \\ & \text{true} & : f_i \end{pmatrix} \longrightarrow f_1 = \begin{pmatrix} \text{case} & a = b : f_1 \\ & \text{true} : f_2 \end{pmatrix}$$

- Original formula:

$$a = b \rightarrow F(G(a) = F(G(b))$$

Example of Bryant's reduction

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- Original formula:

$$a = b \rightarrow F(G(a) = F(G(b))$$

- Number the instances:

$$a = b \rightarrow F_1(G_1(a) = F_2(G_2(b))$$

- Replace each function application with an expression:

$$a = b \rightarrow F_1^* = F_2^*$$

where

$$
\begin{aligned}
F_1^* &= f_1 \\
F_2^* &= \left(\begin{array}{ll} \text{case} & G_1^* = G_2^* \ : f_1 \\ & \text{true} \quad\quad : f_2 \end{array} \right) \\
G_1^* &= g_1 \\
G_2^* &= \left(\begin{array}{ll} \text{case} & a = b \ : g_1 \\ & \text{true} \quad : g_2 \end{array} \right)
\end{aligned}
$$

- Uninterpreted functions give us the ability to represent an *abstract* view of functions.
- It over-approximates the concrete system.

    $1 + 1 = 1$ is a contradiction

    But

    $F(1, 1) = 1$ is satisfiable!

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  But

  $F(1, 1) = 1$ is satisfiable!

- Conclusion: unless we are careful, we can give wrong answers, and this way, loose soundness.

- In general, a sound but incomplete method is more useful than an unsound but complete method.

- A sound but incomplete algorithm for deciding a formula with uninterpreted functions φ^{UF}:

    1. Transform it into Equality Logic formula φ^E
    2. If φ^E is unsatisfiable, return 'Unsatisfiable'
    3. Else return 'Don't know'

- Question #1: is this useful?

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- When the abstract view is sufficient for the proof, it enables (or at least simplifies) a mechanical proof.
- So when is the abstract view sufficient?

- (common) Proving equivalence between:
    - Two versions of a hardware design (one with and one without a pipeline)
    - Source and target of a compiler ("Translation Validation")

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  - Two versions of a hardware design (one with and one without a pipeline)
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- (rare) Proving properties that do not rely on the exact functionality of some of the functions

- Assume the source program has the statement

$$z = (x_1 + y_1) \cdot (x_2 + y_2);$$

which the compiler turned into:

$$u_1 = x_1 + y_1;$$
$$u_2 = x_2 + y_2;$$
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- We need to prove that:

$$(u_1 = x_1 + y_1 \quad \wedge \quad u_2 = x_2 + y_2 \quad \wedge \quad z = u_1 \cdot u_2)$$
$$\longrightarrow \quad (z = (x_1 + y_1) \cdot (x_2 + y_2))$$

Example: Translation Validation

- Claim: φ^{UF} is valid

- We will prove this by reducing it to an Equality Logic formula

$$\varphi^E = \left(\begin{array}{c} (x_1 = x_2 \wedge y_1 = y_2 \longrightarrow f_1 = f_2) \wedge \\ (u_1 = f_1 \wedge u_2 = f_2 \longrightarrow g_1 = g_2) \end{array} \right) \longrightarrow$$
$$((u_1 = f_1 \wedge u_2 = f_2 \wedge z = g_1) \longrightarrow z = g_2)$$

- Good: each function on the left can be mapped to a function on the right with equivalent arguments

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- Bad: almost all other cases

- Example:

$$\underline{\text{Left}} \qquad \underline{\text{Right}}$$
$$x + x \qquad 2x$$

- This is easy to prove:

$$(x_1 = x_2 \wedge y_1 = y_2) \longrightarrow (x_1 + y_1 = x_2 + y_2)$$

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- Fix by adding:

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- What about *other cases*?
  Use more rewriting rules!

```
int power3(int in) {
    out = in;
                                    int power3_new(int in) {
    for(i=0; i<2; i++)                 out = (in*in)*in;
       out = out * in;                 return out;
                                    }
    return out;
}
```

- These two functions return the same value regardless if it is '*' or any other function.
- *Conclusion*: we can prove equivalence by replacing '*' with an uninterpreted function

- But first we need to know how to turn programs into equations.
- There are several options – we will see static single assignment for bounded programs.

Static Single Assignment (SSA) form

- $\rightarrow$ see compiler class
- Idea: Rename variables such that each variable is assigned exactly once

Example:

`x=x+y;`	$x_1 = x_0 + y_0;$
`x=x*2;`	$x_2 = x_1 * 2;$
`a[i]=100;`	$a_1[i_0] = 100;$

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- Idea: Rename variables such that each variable is assigned exactly once

  Example:
  ```
  x=x+y;
  x=x*2;
  a[i]=100;
  ```
  $\Rightarrow$
  ```
  x₁=x₀+y₀;
  x₂=x₁*2;
  a₁[i₀]=100;
  ```

- Read assignments as equalities
- Generate constraints by simply conjoining these equalities

  Example:
  ```
  x₁=x₀+y₀;
  x₂=x₁*2;
  a₁[i₀]=100;
  ```
  $\Rightarrow$
  $$x_1 = x_0 + y_0 \quad \wedge$$
  $$x_2 = x_1 * 2 \quad \wedge$$
  $$a_1[i_0] = 100$$

What about if? Branches are handled using ϕ-nodes.

```
int main() {
   int x, y, z;

   y=8;

   if(x)
     y--;
   else
     y++;


   z=y+1;
}
```

What about `if`? Branches are handled using ϕ-nodes.

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int main() {
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     y++;


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}
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```
int main() {
   int x, y, z;

   y_1=8;

   if(x_0)
     y_2=y_1-1;
   else
     y_3=y_1+1;

   y_4=\phi(y_2, y_3);

   z_1=y_4+1;
}
```

What about `if`? Branches are handled using ϕ-nodes.

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int main() {
   int x, y, z;

   y=8;

   if(x)
     y--;
   else
     y++;


   z=y+1;
}
```

```
int main() {
   int x, y, z;

   y₁=8;

   if(x₀)
     y₂=y₁-1;
   else
     y₃=y₁+1;

   y₄=φ(y₂, y₃);

   z₁=y₄+1;
}
```

$$y_1 = 8 \qquad \wedge$$
$$y_2 = y_1 - 1 \qquad \wedge$$
$$y_3 = y_1 + 1 \qquad \wedge$$
$$y_4 =$$
$$(x_0 \neq 0 \,?\, y_2 : y_3) \wedge$$
$$z_1 = y_4 + 1$$

What about loops?
  → We unwind them!

```
void f(...)  {
  ...
  while(cond) {
   BODY;
  }
  ...
  Remainder;
}
```

SSA for bounded programs

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Some caveats:

- Unwind how many times?
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There is a tool available that does this

- CBMC – C Bounded Model Checker
- Bound is verified using unwinding assertions
- Used frequently for embedded software
  $\longrightarrow$ Bound is a run-time guarantee
- Integrated into Eclipse
- Decision problem can be exported

```
int power3(int in) {
    out = in;

    for(i=0; i<2; i++)
        out = out * in;

    return out;
}
```

```
int power3_new(int in) {
    out = (in*in)*in;
    return out;
}
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       out = out * in;                  return out;
                                    }
    return out;
}
```

Static single assignment (SSA) form:

$out_1 = in \land$
$out_2 = out_1 * in \land$ $out_1' = (in * in) * in$
$out_3 = out_2 * in$

Prove that both functions return the same value:

$$out_3 = out_1'$$

Static single assignment (SSA) form:

$out_1 = in \, \wedge$

$out_2 = out_1 * in \, \wedge$ $\qquad\qquad out'_1 = (in * in) * in$

$out_3 = out_2 * in$

With uninterpreted functions:

$out_1 = in \, \wedge$

$out_2 = F(out_1, in) \, \wedge$ $\qquad\qquad out'_1 = F(F(in, in), in)$

$out_3 = F(out_2, in)$

Static single assignment (SSA) form:

$$out_1 = in \wedge$$
$$out_2 = out_1 * in \wedge \qquad\qquad out_1' = (in * in) * in$$
$$out_3 = out_2 * in$$

With uninterpreted functions:

$$out_1 = in \wedge$$
$$out_2 = F(out_1, in) \wedge \qquad\qquad out_1' = F(F(in, in), in)$$
$$out_3 = F(out_2, in)$$

With numbered uninterpreted functions:

$$out_1 = in \wedge$$
$$out_2 = F_1(out_1, in) \wedge \qquad\qquad out_1' = F_4(F_3(in, in), in)$$
$$out_3 = F_2(out_2, in)$$

With numbered uninterpreted functions:

$$out_1 = in \,\wedge$$
$$out_2 = F_1(out_1, in) \,\wedge \qquad\qquad out_1' = F_4(F_3(in, in), in)$$
$$out_3 = F_2(out_2, in)$$

Example: equivalence of C programs (4/4)

With numbered uninterpreted functions:

$out_1 = in \wedge$
$out_2 = F_1(out_1, in) \wedge$ $out'_1 = F_4(F_3(in, in), in)$
$out_3 = F_2(out_2, in)$

Ackermann's reduction:

$out_1 = in \wedge$
$\varphi_a^E : \quad out_2 = f_1 \wedge$ $\varphi_b^E : \ out'_1 = f_4$
$out_3 = f_2$

Example: equivalence of C programs (4/4)

With numbered uninterpreted functions:
$$out_1 = in \wedge$$
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$$out_3 = F_2(out_2, in)$$

Ackermann's reduction:
$$\varphi_a^E : \quad \begin{aligned} out_1 &= in \wedge \\ out_2 &= f_1 \wedge \\ out_3 &= f_2 \end{aligned} \qquad\qquad \varphi_b^E : out'_1 = f_4$$

The verification condition:

$$\left[\begin{pmatrix} (out_1 = out_2 \rightarrow f_1 = f_2) & \wedge \\ (out_1 = in & \rightarrow f_1 = f_3) & \wedge \\ (out_1 = f_3 & \rightarrow f_1 = f_4) & \wedge \\ (out_2 = in & \rightarrow f_2 = f_3) & \wedge \\ (out_2 = f_3 & \rightarrow f_2 = f_3) & \wedge \\ (in = f_3 & \rightarrow f_3 = f_4) \end{pmatrix} \wedge \varphi_a^E \wedge \varphi_b^E \right] \longrightarrow out_3 = out'_1$$

- Let n be the number of instances of $F()$
- Both reduction schemes require $O(n^2)$ comparisons
- This can be the *bottleneck* of the verification effort

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- Both reduction schemes require $O(n^2)$ comparisons
- This can be the *bottleneck* of the verification effort

- Solution: try to *guess* the pairing of functions
- Still sound: wrong guess can only make a valid formula invalid

- Given $x_1 = x_1'$, $x_2 = x_2'$, $x_3 = x_3'$, prove $\models o_1 = o_2$.

$$o_1 = (\underbrace{x_1 + (a \cdot x_2)}_{f_1}) \ \wedge \ a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = (\underbrace{x_1' + (b \cdot x_2')}_{f_3}) \ \wedge \ b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$

- 4 function instances $\rightarrow$ 6 comparisons

- Given $x_1 = x_1'$, $x_2 = x_2'$, $x_3 = x_3'$, prove $\models o_1 = o_2$.

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- 4 function instances $\rightarrow$ 6 comparisons
- Guess: validity does not rely on $f_1 = f_2$ or on $f_3 = f_4$
- Idea: only enforce functional consistency of pairs (Left,Right).

$$o_1 = \underbrace{(x_1 + (a \cdot x_2))}_{f_1} \wedge a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = \underbrace{(x_1' + (b \cdot x_2'))}_{f_3} \wedge b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$

- Down to 4 comparisons!

$$o_1 = \underbrace{(x_1 + (a \cdot x_2))}_{f_1} \wedge a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = \underbrace{(x_1' + (b \cdot x_2'))}_{f_3} \wedge b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$

- Down to 4 comparisons!
- Another guess: equivalence only depends on $f_1 = f_3$ and $f_2 = f_4$
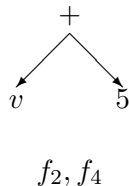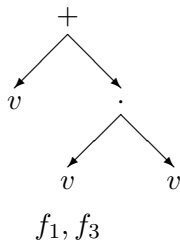- *Pattern matching* may help here

$$o_1 = (\underbrace{x_1 + (a \cdot x_2)}_{f_1}) \,\wedge\, a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = (\underbrace{x_1' + (b \cdot x_2')}_{f_3}) \,\wedge\, b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$

Match according
to patterns
('signatures')

Down to 2 comparisons!
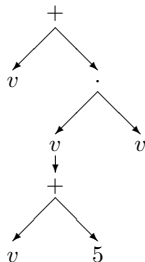
f_1, f_3 $\qquad$ f_2, f_4

$$o_1 = \underbrace{(x_1 + (a \cdot x_2))}_{f_1} \wedge a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = \underbrace{(x_1' + (b \cdot x_2'))}_{f_3} \wedge b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$
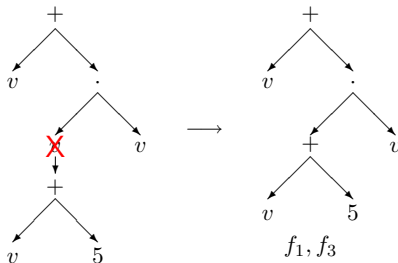
Substitute
intermediate
variables (in the
example: a, b)

$$o_1 = (\underbrace{x_1 + (a \cdot x_2)}_{f_1}) \,\wedge\, a = \underbrace{x_3 + 5}_{f_2} \qquad \text{Left}$$

$$o_2 = (\underbrace{x_1' + (b \cdot x_2')}_{f_3}) \,\wedge\, b = \underbrace{x_3' + 5}_{f_4} \qquad \text{Right}$$

Substitute intermediate variables (in the example: a, b)

The SSA example revisited (1)

With numbered uninterpreted functions:

$$out_1 = in \wedge$$
$$out_2 = F_1(out_1, in) \wedge \qquad\qquad out_1' = F_4(F_3(in, in), in)$$
$$out_3 = F_2(out_2, in)$$

The SSA example revisited (1)

With numbered uninterpreted functions:

$out_1 = in \wedge$
$out_2 = F_1(out_1, in) \wedge$ $out'_1 = F_4(F_3(in, in), in)$
$out_3 = F_2(out_2, in)$

Map F_1 to F_3:

Map F_2 to F_4:

The SSA example revisited (2)
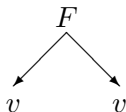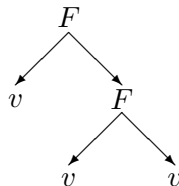
With numbered uninterpreted functions:

$$out_1 = in \wedge$$
$$out_2 = F_1(out_1, in) \wedge \qquad\qquad out'_1 = F_4(F_3(in, in), in)$$
$$out_3 = F_2(out_2, in)$$

Ackermann's reduction:

$$out_1 = in \wedge$$
$$\varphi_a^E : \quad out_2 = f_1 \wedge \qquad\qquad \varphi_b^E : out'_1 = f_4$$
$$out_3 = f_2$$

The verification condition has *shrunk*:

$$\left[\left(\begin{array}{c} (out_1 = in \longrightarrow f_1 = f_3) \quad \wedge \\ (out_2 = f_3 \longrightarrow f_2 = f_4) \end{array} \right) \wedge \varphi_a^E \wedge \varphi_b^E \right] \longrightarrow out_3 = out'_1$$

Same example with Bryant's reduction

With numbered uninterpreted functions:

$$out_1 = in \land$$
$$out_2 = F_1(out_1, in) \land \qquad out_1' = F_4(F_3(in, in), in)$$
$$out_3 = F_2(out_2, in)$$

Bryant's reduction:

$$\varphi_a^E : \begin{array}{l} out_1 = in \land \\ out_2 = f_1 \land \\ out_3 = f_2 \end{array}$$

$$\varphi_b^E : out_1' = \left(\text{case} \ \begin{array}{ll} \left(\text{case} \ \begin{array}{ll} in = out_1 : f_1 \\ \text{true} \quad : f_3 \end{array} \right) = out_2 : f_2 \\ \text{true} \qquad\qquad\qquad\qquad : f_4 \end{array} \right)$$

The verification condition:

$$(\varphi_a^E \land \varphi_b^E) \longrightarrow out_3 = out_1'$$

1. It is expressible enough to state something interesting.

2. It is decidable and more efficiently solvable than richer logics, for example in which some functions are interpreted.

3. Models which rely on infinite-type variables are expressed more naturally in this logic in comparison with Propositional Logic.