

# A Logic for Correlating Temporal Properties across Program Transformations

Aditya Kanade<sup>1</sup>, Amitabha Sanyal<sup>2</sup>, and Uday P. Khedker<sup>2</sup>

<sup>1</sup>Indian Institute of Science, <sup>2</sup>Indian Institute of Technology Bombay

**Abstract.** Program transformations are widely used in synthesis, optimization, and maintenance of software. Correctness of program transformations depends on preservation of some important properties of the input program. By regarding programs as Kripke structures, many interesting properties of programs can be expressed in temporal logics. In temporal logic, a formula is interpreted on a single program. However, to prove correctness of transformations, we encounter formulae which contain some subformulae interpreted on the input program and some on the transformed program. An example where such a situation arises is verification of optimizing program transformations applied by compilers.

In this paper, we present a logic called Temporal Transformation Logic (TTL) to reason about such formulae. We consider different types of primitive transformations and present TTL inference rules for them. Our definitions of program transformations and temporal logic operators are novel in their use of the boolean matrix algebra. This results in specifications that are succinct and constructive. Further, we use the boolean matrix algebra in a uniform manner to prove soundness of the TTL inference rules.

**Keywords:** Program transformations; Temporal logic; Compiler verification; Boolean matrix algebra

## 1. Introduction

Program transformations are widely used in synthesis, optimization, and maintenance of software. Correctness of program transformations depends on preservation of some important properties of the input program. By regarding programs as Kripke structures, many interesting properties of programs can be expressed in temporal logics [Pnu77]. In temporal logic, a formula is interpreted on a single program. However, to prove correctness of transformations, we encounter formulae which contain some subformulae interpreted on the input program and some on the transformed program. The proof systems for temporal logics [MP91, PK02] are therefore not sufficient to prove correlations of temporal properties across program transformations. In this paper, we present a logic called Temporal Transformation Logic (TTL) to reason about such formulae.

Our study of this logic is motivated by an interesting application in the area of compiler verification.

A compiler optimizer analyzes a program and identifies potential performance improvements. An optimized version of the program is then obtained by applying several transformations to the input program. However, a mistake in the design of an optimizer can proliferate in the form of bugs in the softwares compiled through it. It is therefore important to verify whether an optimization routine preserves program semantics. Proving semantic equivalence of programs is usually tedious. The complexity of proofs can be conquered by taking advantage of the fact that optimizations with similar objectives employ similar program transformations. This observation lead to identification of transformation primitives and their soundness conditions [KSK07].

A transformation primitive denotes a small-step program transformation that is used in many optimizing transformations. These primitives can be used to specify a large class of optimizations by sequential composition. The soundness condition for a transformation primitive is a condition on the input programs to the primitive which if satisfied implies that the transformed program is semantically equivalent to the input program. This approach reduces proving soundness of an optimization to merely showing that soundness conditions of the underlying primitives are satisfied on the versions of the input program on which they are applied. This is much simpler than directly proving semantics preservation for each optimization.

Program analyses that are used for identifying optimization opportunities and the soundness conditions of the transformation primitives are global dataflow properties. These properties can be either forward, backward, or more generally bidirectional [DK93]. Further, these can be classified as may or must properties depending on whether they are defined in terms of some or all paths originating (for forward analyses) or terminating (for backward analyses) at a node. These properties can be expressed naturally in a temporal logic called computational tree logic with branching past ( $CTL_{bp}$ ) [KP95].  $CTL_{bp}$  formulae are state formulae and thus application points for program transformations can be identified by models of  $CTL_{bp}$  formulae.

A program analysis used in an optimization is interpreted on the input program. However, the input program is transformed step-by-step and hence the soundness condition of a transformation primitive is interpreted on the version of the input program to which it is applied. Thus to prove soundness of an optimization, we need a logic to correlate temporal properties of programs across transformations. Program transformations are used widely in various software engineering activities like program synthesis, refactoring, software renovation, and reverse engineering [Vis01]. Temporal logic is known to be a powerful language for specifying properties of program executions. We therefore believe that TTL will also be useful in verification problems arising in these domains.

A program transformation may involve changes to the control flow graph (structural transformation) and to the program statements (content transformation). In general, if we consider arbitrary big-step transformations resulting in diverse changes to the input program, we may not be able to correlate any interesting temporal properties across them. We therefore present inference rules for the (small-step) transformation primitives. A larger transformation is expressed as a sequential composition of such primitives. A correlation between temporal properties of its input and output programs can be inferred by using the rules for the component transformations on intermediate versions of the program.

The semantics of temporal operators is usually defined in a relational setting by considering transitive closure (paths) of the transition relation (edges) of a Kripke structure. We make novel use of the boolean matrix algebra to define temporal logic operators and also the primitive program transformations. The definitions are succinct and facilitate algebraic proofs of soundness of the TTL inference rules. Further, the definitions are constructive and can be evaluated directly. The choice of boolean matrix algebra thus enables both verification and translation validation of compiler optimizations [KSK06, KSK07, KKS].

## Related work

Temporal logic has been used for specifying dataflow properties. In [Sch98, SS98], Schmidt et al. propose that static analysis of a program can be viewed as model checking of an abstraction of the program, with the properties of interest specified in a suitable temporal logic. In the recent literature on verification of compiler optimizations [LJWF02, Fre02, LMC03], conditional rewrites are proposed as a means for specifying optimizing transformations. The enabling condition of a rewrite is defined in a temporal logic. However, unlike our approach [KSK07], they do not use the temporal logic in verification. The verification is performed by proving semantic equivalence of programs by induction on length of program executions.

Kripke structures serve as a natural modeling paradigm for system specification when the properties of interest are temporal in nature. Simulation relations [Mil71, Nam97] make it possible to correlate temporal properties between Kripke structures and are used to show refinement or equivalence of system specifications.

Transformations of Kripke structures arise in state space reduction techniques [Par81, FV99, BdS93, HHK95] where a Kripke structure is transformed to a (bi)similar Kripke structure with less number of states. Model checking certain formulae over the reduced Kripke structure is equivalent to model checking the formulae on the original Kripke structure. In order to prove soundness of the TTL inference rules, we also construct simulation relations between programs (seen as Kripke structures). However, unlike the usual set-based formulation of simulation relations, we present boolean matrix algebraic formulation. This enables us to prove correlations of temporal properties between programs (Kripke structures) in a completely novel boolean matrix algebraic setting.

Boolean matrix algebra or more generally, modal algebra, has been used for defining semantics of modal logic operators [Tho72, BdRV01]. The boolean algebra is used for modeling propositional logic and the boolean matrix algebraic operators are used for capturing different modalities. In [Fit03], Fitting develops a modal algebraic formulation of bisimulation in modal and multi-modal setting. Our definitions correspond to frame bisimulations in [Fit03]. However, our approach is constructive, that is, for each type of primitive transformations, we show that a simulation relation of a certain nature exists between a program and its transformed version under the primitive transformation. To the best of our knowledge, our definitions of program transformations and proofs of soundness of correlations of temporal properties across transformations is the first approach which makes use of the boolean matrix algebra in this setting.

Temporal logic model checking is an algorithmic technique of checking whether a model satisfies a temporal logic formula (cf. [CGP00]). The validation of individual program transformations can be achieved by model checking a specified property on the input program and (with possible renaming of atomic propositions) also on its transformed version. For translation validation of compiler optimizers, model checking can be used effectively [KSK06, KKS]. In this paper, we address the problem of proving preservation of temporal properties for all possible applications of a transformation primitive. In temporal logic, a formula is interpreted on a single program and therefore the proof systems for temporal logics [MP91, PK02] are not sufficient in this setting.

## Outline

In Section 2, we motivate the need for a logic like TTL through an application to verification of compiler optimizations. In Section 3, we define several primitive graph transformations which are used in Section 4 to define primitive program transformations. In Section 5, we define the notion of transformations of Kripke structures and model program transformations as transformations of Kripke structures. The syntax, semantics, and inference rules of TTL are presented in Section 6. The proofs of soundness of the TTL inference rules are derived in Section 7. In Section 8, we present the conclusions.

## 2. Motivation: Verification of compiler optimizations

### Specification scheme

Consider the formal specification of common subexpression elimination (CSE) optimization given in Figure 1. We use PVS (Prototype Verification System) [OSRSC99] to develop, validate, and verify formal specifications of optimizations. In this paper, we explain the CSE specification intuitively without giving details of the PVS specification language. The specification defines both, the program analysis and the optimizing transformation. In Figure 1, `prog` is a program and `e` is an expression in `prog`. A function that defines a program analysis returns a boolean vector (whose type is a column matrix *mat*) such that the boolean value corresponding to a program point denotes whether the property holds at the program point or not. We explain the CSE specification with an example optimization shown in Figure 2.

The functions `Transp`, `Antloc`, and `Comp` define *local dataflow properties* i.e. the dataflow information that depends only on the statement at a program point. An expression `e` is `Transp` at a program point if none of its operands are modified at the program point. An expression `e` is `Antloc` at a program point if the expression is computed at the program point. If an expression is both `Transp` and `Antloc` at a program point then it is `Comp` at the program point. For brevity, we do not give the definitions of these local dataflow properties. For the definitions, we refer the reader to [Kan07]. For `prog1` in Figure 2, the expression `a/b` is `Transp` at all the program points, and is `Antloc` and `Comp` at program points 2, 3, and 5.

```

Avail(prog, e) : mat    = Comp(prog, e)
Redund(prog, e) : mat  = Antloc(prog, e) * AY(prog'cfg, AS(prog'cfg, Transp(prog, e), Avail(prog, e)))
OrgAvail(prog, e) : mat = Avail(prog, e) - Redund(prog, e)

CSE_Transformation(prog1, e): Program =
  IF member(e, Expressions(prog1)) ∧ ¬ BASE?(e) THEN
    LET n1      = length(prog1'cfg'T),
        orgavails = OrgAvail(prog1, e),
        m      = Count1s(orgavails),
        redund  = Redund(prog1, e),
        newpoints = append(zerol(n1), onel(m)),
        prog2   = IP(prog1, orgavails, newpoints),
        t      = NEWVAR(prog2),
        prog3   = IA(prog2, newpoints, t, e),
        orgavails3 = append(orgavails, zerol(m)),
        prog4   = RE(prog3, orgavails3, e, t),
        redund4 = append(redund, zerol(m))
    IN
  ELSE prog1 ENDIF

```

Fig. 1. A specification of common subexpression elimination optimization

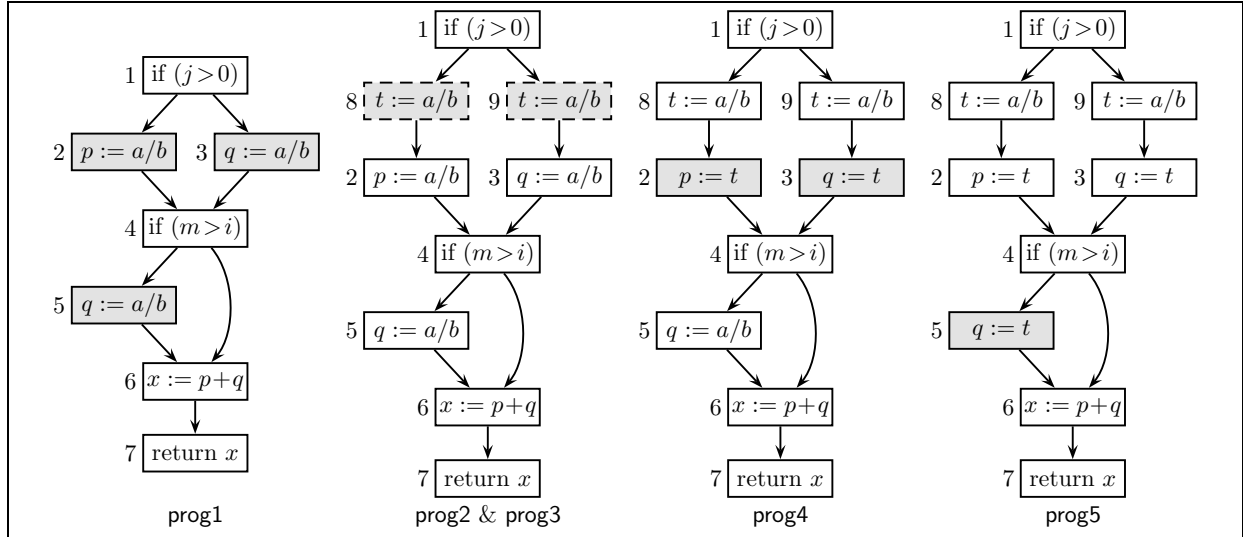


Fig. 2. An example of common subexpression elimination optimization

The function `Redund` defines a *global dataflow property* i.e. the dataflow information that depends on the statements along the paths ending or starting at a program point. We specify global dataflow properties using  $\text{CTL}_{\text{bp}}$ . The functions `AY` and `AS` respectively denote the *universal predecessor* and the *universal since*  $\text{CTL}_{\text{bp}}$  operators. Since an optimization transforms a program step-by-step, we have different versions of the input program. Thus, in the PVS specifications, we pass the control flow graph of a program (denoted as `prog'cfg`) that a  $\text{CTL}_{\text{bp}}$  operator is interpreted on as a parameter to the function. However, in the latter part of the paper, we follow the usual convention where temporal operators do not take a model as a parameter.

A program point satisfies the `Redund` property if (1) the expression `e` is `Antloc` at a program point and (2) from all its predecessors, the expression is `Transp` along all incoming paths, since it is `Avail`. The origins of availability (`OrgAvail`) are the program points where the expression is `Comp` but not `Redund`. For `prog1` in Figure 2, the expression `a/b` is `Redund` at program point 5 and it is `OrgAvail` at program points 2 and 3.

The function `CSE_Transformation` is an *optimizing transformation* i.e. it takes an input program and possi-

bly other parameters, and applies several primitive transformations to the input program. *CSE.Transformation* is defined as the following sequence of transformations:

1. Transform *prog1* to *prog2*: Insert a predecessor (IP) each to the program points that satisfy the *OrgAvail* property in *prog1*. In Figure 2, program points 8 and 9 are the new predecessors to program points 2 and 3. Due to space constraints, *prog2* and *prog3* (a transformed version of *prog2*) are represented as same programs in Figure 2. In *prog2*, program points 2 and 3 contain skip statements. The statements shown at program points 2 and 3 are obtained by the next transformation.
2. Transform *prog2* to *prog3*: Let *t* be a new variable. Insert an assignment (IA)  $t := e$  at the new program points introduced in the first transformation. In Figure 2, the assignment  $t := a/b$  (where *t* is a new variable and  $a/b$  is the expression under consideration) is inserted at program points 8 and 9.
3. Transform *prog3* to *prog4*: Replace the occurrences of *e* at the program points that satisfy the *OrgAvail* property in *prog1* by *t*. In Figure 2, the computations of  $a/b$  at program points 2 and 3 are replaced by *t*.
4. Transform *prog4* to *prog5*: Replace the occurrences of the expression (RE) *e* at the program points that satisfy the *Redund* property in *prog1* by *t*. In Figure 2, the computation of  $a/b$  at program point 5 is replaced by *t*.

In Section 4, we explain the definitions of the transformation primitives: insertion of predecessors (IP), insertion of assignments (IA), and replacement of expressions (RE). An extensive set of transformation primitives is defined in [Kan07].

## Soundness conditions

Consider an application of  $RE : RE(\text{prog}, \text{points}, e, v)$ . It returns a transformed version of the input program *prog* where the occurrences of the expression *e* at the program points denoted by *points* are replaced by a variable *v*. The replacement of an occurrence of *e* at a program point *p* by *v* preserves semantics if *v* and *e* have same value just before *p*. The soundness conditions for RE are as follows:

- (1) The statement at *p* is an assignment statement. *AND*
- (2) The variable *v* is not an operand of the expression *e*. *AND*
- (3) The expression *e* is computed at *p* (*Antloc*). *AND*
- (4) Depending on the nature of *e*, we have the following conditions:
  - (a) If the expression is a just a variable (say *x*) then along each backward path starting from the predecessors of *p* one of the following should hold:
    - An assignment  $v := x$  or an assignment  $x := v$  is encountered and in between there is no other assignment to any of the variables *v* or *x*.
    - The assignments  $v := h$  and  $x := h$  for some expression *h* are encountered (in some order) at program points say  $p_1$  and  $p_2$ . In between  $p_1$  and  $p_2$ , the expression *h* is *Transp* ensuring that both *v* and *x* has the same value. Upto the first of the program points  $p_1$  and  $p_2$ , along the backward path from *p*, neither *v* nor *x* is assigned.
  - (b) Otherwise, along each backward path starting from the predecessors of *p*, an assignment  $v := e$  is encountered with no assignment to *v* or the variable operands of *e* in between.

Similarly soundness conditions for other transformation primitives can also be defined. It can be observed that the soundness conditions can be naturally expressed in  $CTL_{bp}$ . For each transformation primitive, we identify the soundness conditions and separately prove that if the soundness conditions are satisfied by an application of the primitive then the transformed program is semantically equivalent to the input program. These are one-time proofs. Since the primitives are small-step transformations, the proofs of semantics preservation for them are easier than similar proofs for (large-step) optimizing transformations.

## Verification scheme

Consider an optimizing transformation  $T$  applied to an input program  $P_1$ :

$$T(P_1) \triangleq \text{LET } P_2 = T_1(P_1, \pi_1), \dots, P_k = T_{k-1}(P_{k-1}, \pi_{k-1}) \text{ IN } T_k(P_k, \pi_k)$$

where a transformation primitive  $T_i$  is applied to a program  $P_i$  at program points  $\pi_i$ . Let other parameters of the transformation primitives be implicit.

Let  $\varphi_1, \dots, \varphi_k$  respectively be the soundness conditions of the primitives  $T_1, \dots, T_k$ . The soundness of  $T$  can be established by proving that for each  $i \in \{1, \dots, k\}$ , the verification condition  $\varphi_i(P_i, \pi_i)$  is satisfied. Thus, proving soundness of an optimization reduces to showing that the soundness conditions of the underlying transformation primitives are satisfied. Optimizations with similar objectives comprise similar transformations. For example, “replacement of some occurrences of an expression by a variable” is a transformation which is common to optimizations like common subexpression elimination, lazy code motion, loop invariant code motion, and several others whose aim is to avoid unnecessary recomputations of a value. Thus identification of transformation primitives and their soundness conditions simplifies both specification and verification of a class of optimizations.

To prove a verification condition  $\varphi_i(P_i, \pi_i)$ , we use the program analysis and properties of the preceding transformations. However, the program analysis is interpreted on the input program and properties of the preceding transformations are interpreted on previous versions of the program. Thus we require a logic to correlate temporal properties across transformations of programs. We therefore develop the temporal transformation logic in this paper.

## 3. Primitive graph transformations

A structural transformation of a program involves a transformation of the control flow graph of the program. For example, the insertion of predecessors (IP) transformation used in the CSE specification (Figure 1) is a structural transformation. In this section, we define several primitive graph transformations which are then used, in the next section, to define primitive program transformations.

### 3.1. Notation

We use the boolean matrix algebra in a novel way to define graph transformations. We now introduce the notation and terminology for boolean matrix algebraic operations and graph transformations.

#### *Boolean matrix algebra*

Let  $\mathcal{B} = \{0, 1\}$  be the set of boolean values. Consider a boolean algebra  $\mathcal{A} = (\mathcal{B}, \neg, \vee, \wedge, 0, 1)$  where  $\neg, \vee,$  and  $\wedge$  are respectively the boolean negation, disjunction, and conjunction operators. Let 0 and 1 respectively be the identity elements of  $\vee$  and  $\wedge$  operators.

For any two positive natural numbers  $n$  and  $m$ , consider a function

$$\mathcal{B}_{n,m} : \{1, \dots, n\} \times \{1, \dots, m\} \rightarrow \mathcal{B}.$$

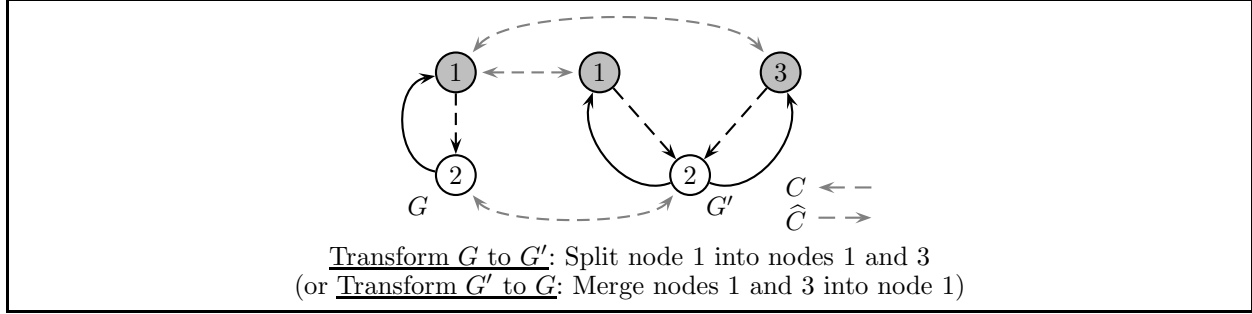
Let us consider  $\mathcal{B}_{n,m}$  as an  $(n \times m)$  boolean matrix with  $n$  rows and  $m$  columns. For a matrix  $U$ , let  $[U]_i^j$  denote the element at the  $i$ th row and the  $j$ th column. Let  $[U]_i$  denotes the  $i$ th row vector of  $U$  and  $[U]^j$  denote the  $j$ th column vector of  $U$ .

Consider a boolean matrix algebra  $\mathcal{A}_{n,m}$  defined as follows:

$$\mathcal{A}_{n,m} = \left( \mathcal{B}_{n,m}, \overline{\phantom{x}}, \widehat{\phantom{x}}, +, *, \cdot, -, \mathbf{0}_{n,m}, \mathbf{1}_{n,m} \right)$$

The functions in the algebra are defined as follows:

1.  $\overline{U}$  denotes the negation of  $U$  which extends the boolean negation to matrices.
2.  $\widehat{U}$  denotes the transpose of  $U$ .



**Fig. 3.** An Example of Node Splitting (or Node Merging)

3. If  $U$  and  $V$  are two  $(n \times m)$  matrices, the addition “ $U+V$ ”, the product “ $U*V$ ”, and the subtraction “ $U-V$ ” are defined as follows:

$$[U+V]_i^j \triangleq [U]_i^j \vee [V]_i^j \quad [U*V]_i^j \triangleq [U]_i^j \wedge [V]_i^j \quad [U-V]_i^j \triangleq [U]_i^j \wedge \neg[V]_i^j$$

4. If  $U$  is an  $(n \times m)$  matrix and  $V$  is an  $(m \times p)$  matrix, the multiplication “ $U \cdot V$ ” gives an  $(n \times p)$  matrix as follows:

$$[U \cdot V]_i^j \triangleq \bigvee_{1 \leq k \leq m} ([U]_i^k \wedge [V]_k^j)$$

$\mathbf{0}_{n,m}$  is an  $(n \times m)$  matrix whose all elements are 0s and is the identity element of the operator  $+$ .  $\mathbf{1}_{n,m}$  is an  $(n \times m)$  matrix whose all elements are 1s and is the identity element of the operator  $*$ .

For readability, we do not explicate dimensions of matrices and assume appropriate dimensions as per their usage. We use bold letters to denote boolean vectors. Boolean vectors are also considered as column matrices. The two special vectors  $\vec{0}$  and  $\vec{1}$  have all the elements as 0s and 1s respectively.

The precedence of the operators in the decreasing order is as follows:  $[\neg \wedge \vee][\cdot][+ - *]$ . The operators within a pair of brackets have the same precedence and their usage needs to be disambiguated with appropriate parenthesization.

### Graph transformations

A graph  $G$  is specified by a set of nodes  $N$  and a boolean adjacency matrix  $A$  representing the edges of  $G$ . We assume an implicit ordering of nodes to access elements in matrices and vectors associated with a graph. The ordering is considered to be constant for a graph.

Consider a graph  $G = (N, A)$  and its transformed version  $G' = (N', A')$ . A *correspondence relation*  $C$  for a transformation of  $G$  to  $G'$  gives the correspondence between the nodes of  $G'$  and the nodes of  $G$ . It is denoted as an  $(|N'| \times |N|)$  boolean matrix. We define the transformation of  $G$  to  $G'$  in terms of (1) the nature of the correspondence relation  $C$  and (2) an equation expressing the adjacency matrix  $A'$  of the transformed graph in terms the adjacency matrix  $A$  of the input graph.

Below we describe seven types of primitive graph transformations. These transformations are orthogonal with each other i.e. none of them can be expressed in terms of the others.

### 3.2. Node Splitting

A node splitting transformation of a graph  $G$  splits at least one node of  $G$  into multiple nodes. The edges associated with a node being split are translated into similar edges for the corresponding nodes in the transformed graph  $G'$ . All the other edges of  $G$  are preserved. For example, consider the graphs  $G$  and  $G'$  shown in Figure 3. Node 1 of  $G$  is split into nodes 1 and 3 to get graph  $G'$ . Edge  $\langle 1, 2 \rangle$  of  $G$  is translated into edges  $\langle 1, 2 \rangle$  and  $\langle 3, 2 \rangle$  of  $G'$ . Similarly, edge  $\langle 2, 1 \rangle$  of  $G$  is translated into edges  $\langle 2, 1 \rangle$  and  $\langle 2, 3 \rangle$  of  $G'$ .

In Figure 3, the correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is shown by the dashed gray arrows from right to left. A correspondence relation arrow from a node  $p$  to a node  $q$  means that the element

at the  $p$ th row and  $q$ th column in the correspondence matrix  $C$  is (boolean) 1 i.e.  $[C]_p^q = 1$ . The matrix  $\widehat{C}$  is represented by the dashed gray arrows from left to right.

The correspondence between the edges of the two graphs can be *traced diagrammatically*. For example, suppose we go from node 3 of  $G'$  to node 1 of  $G$  by following a  $C$  arrow, then we follow edge  $\langle 1, 2 \rangle$  in  $G$ , and lastly, we follow the  $\widehat{C}$  arrow from node 2 of  $G$  to node 2 of  $G'$ . This gives us edge  $\langle 3, 2 \rangle$  in  $G'$ . Similarly, all the other edges of  $G'$  can be traced.

The composition of arrows and edges (which are respectively, relations between the nodes of  $G'$  and  $G$ , and between the nodes of the individual graphs) can be represented very concisely as *matrix multiplications*, when these relations are expressed as *adjacency matrices*. This allows us to express the adjacency matrix of the transformed graph in terms of the adjacency matrix of the input graph.

**Definition 1.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called a *node splitting* transformation if

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a total, onto, and many-to-one relation and
2.  $C \cdot A \cdot \widehat{C} = A'$ .

The first condition specifies the nature of the correspondence relation. The correspondence relation is required to be total and onto. Totality ensures that each node of the transformed graph corresponds to a node of the input graph i.e. no new node is introduced in the transformed graph. The onto nature of the correspondence relation ensures that for each node of the input graph there is a node of the transformed graph i.e. no node of the input graph is deleted. The many-to-one nature of the correspondence relation ensures that at least one node of the transformed graph corresponds to multiple nodes of the input graph i.e. the transformation splits at least one node of the input graph and is not vacuous. We require the non-vacuous nature of the relation so that the definitions of the graph transformations are unambiguous and orthogonal. The second condition defines the adjacency matrix of the transformed graph in terms of the adjacency matrix of the input graph and the correspondence relation.

### 3.3. Node Merging

A node merging transformation of a graph  $G$  merges multiple nodes of  $G$  into a single node. The edges associated with the nodes being merged are coalesced into similar edges for the merged node. All the other edges of  $G$  are preserved. For example, consider the graphs shown in Figure 3 once again. This time consider  $G$  to be a transformation of  $G'$ . Nodes 1 and 3 of  $G'$  are merged into node 1 of  $G$ . The correspondence relation  $C$  is now given by the dashed gray arrows from left to right (opposite to that of the node splitting example). Edges  $\langle 1, 2 \rangle$  and  $\langle 3, 2 \rangle$  of  $G'$  are coalesced into edge  $\langle 1, 2 \rangle$  of  $G$ . Similarly, edges  $\langle 2, 1 \rangle$  and  $\langle 2, 3 \rangle$  of  $G'$  are coalesced into edge  $\langle 2, 1 \rangle$  of  $G$ . As described earlier, the correspondence between the edges of the two graphs can be traced diagrammatically.

**Definition 2.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called a *node merging* transformation if

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a total, onto, and one-to-many relation and
2.  $C \cdot A \cdot \widehat{C} = A'$ .

The first condition ensures that there are no new nodes introduced in the transformed graph (totality) and no nodes of the input graph are deleted (ontones). The one-to-many nature ensures that more than one nodes are merged and the transformation is not vacuous. The second condition defines the adjacency matrix of the graph in terms of the adjacency matrix of the input graph and the correspondence relation.

### 3.4. Edge Addition

An edge addition transformation of a graph  $G$  adds at least one edge to  $G$ . All the edges of  $G$  are preserved. For example, consider the graphs  $G$  and  $G'$  shown in Figure 4 such that  $G'$  is obtained by adding edge  $\langle 1, 3 \rangle$  to



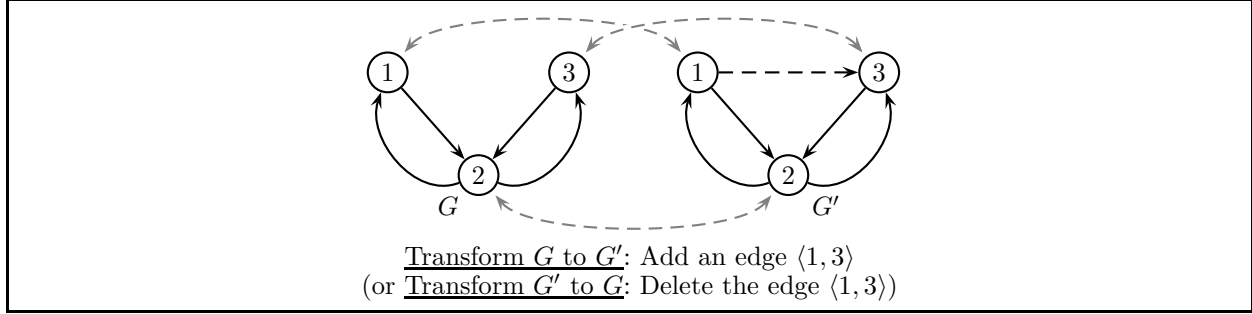


Fig. 4. An Example of Edge Addition (or Edge Deletion)

$G$ . The correspondence relation  $C$  is given by the dashed gray arrows from right to left. The correspondence between the rest of the edges can be traced diagrammatically.

**Definition 3.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called an *edge addition* transformation if

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a bijection and
2. There exists a  $(|N| \times |N|)$  matrix  $E$  s.t.  $A + E > A$  and  $C \cdot A \cdot \widehat{C} + C \cdot E \cdot \widehat{C} = A'$  where the matrix  $E$  denotes the edges to be added to  $G$ .

The first condition ensures that there is a one-to-one correspondence between the nodes of  $G'$  and  $G$ . The second condition ensures that at least one new edge is added.

### 3.5. Edge Deletion

An edge deletion transformation of a graph  $G$  deletes at least one edge of  $G$ . All the other edges of  $G$  are preserved. For example, consider the graphs shown in Figure 4 once again. This time consider  $G$  to be a transformation of  $G'$ . Edge  $\langle 1, 3 \rangle$  is deleted from  $G'$  to get  $G$ . The correspondence relation  $C$  is now given as the dashed gray arrows from left to right (opposite to that of the edge addition example). The correspondence between the rest of the edges can be traced diagrammatically.

**Definition 4.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called an *edge deletion* transformation if

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a bijection and
2. There exists a  $(|N| \times |N|)$  matrix  $E$  s.t.  $A - E < A$  and  $C \cdot A \cdot \widehat{C} - C \cdot E \cdot \widehat{C} = A'$  where the matrix  $E$  denotes the edges to be deleted from  $G$ .

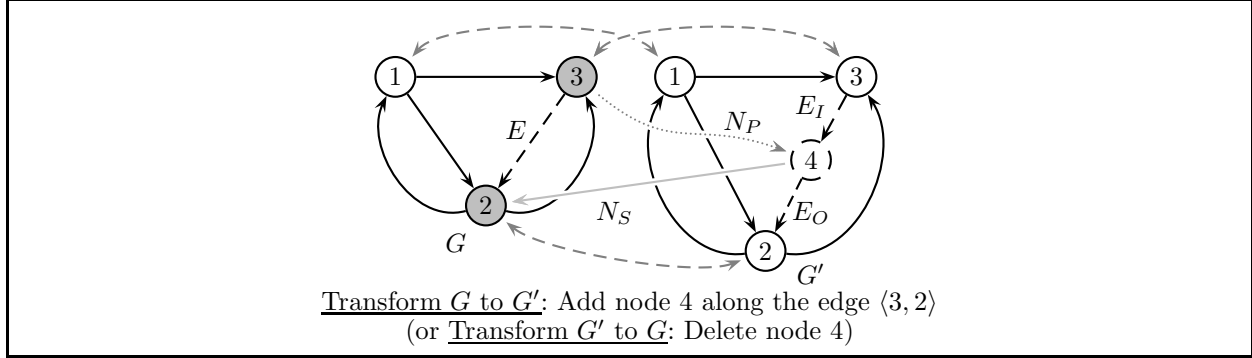
The first condition ensures that there is a one-to-one correspondence between the nodes of  $G'$  and  $G$ . The second condition ensures that at least one edge is deleted.

### 3.6. Node Addition

For a graph  $G$ , given a set of edges  $E$ , a node addition transformation adds a new node along each edge in  $E$ . It splits the edges in  $E$  and adds the new nodes as successors to the source nodes of the edges in  $E$  and as predecessors to the target nodes of the edges in  $E$ . The rest of the edges of  $G$  are preserved.

For example, consider the two graphs shown in Figure 5. The graph  $G'$  is obtained by adding node 4 along edge  $\langle 3, 2 \rangle$  of  $G$ . Edge  $\langle 3, 2 \rangle$  is split into two edges  $\langle 3, 4 \rangle$  and  $\langle 4, 2 \rangle$  making node 4 a successor of node 3 and a predecessor of node 2. The correspondence relation  $C$  is given as the dashed gray arrows from right to left. The lightgray solid arrows from right to left marked as  $N_S$  map the newly added nodes to the target nodes of the edges in  $E$ . The lightgray dotted arrows from left to right marked as  $N_P$  map the source nodes of the edges in  $E$  to the newly added nodes.

The correspondence between the edges of the two graphs can be traced diagrammatically. Edge  $\langle 3, 4 \rangle$  is



**Fig. 5.** An Example of Node Addition (or Node Deletion)

obtained by following the  $C$  arrow from node 3 of  $G'$  to node 3 of  $G$  and then following the  $N_P$  arrow from node 3 to node 4 of  $G'$ . Edge  $\langle 4, 3 \rangle$  is obtained by following the  $N_S$  arrow from node 4 of  $G'$  to node 2 of  $G$  and then following the  $\hat{C}$  arrow from node 2 of  $G$  to node 2 of  $G'$ . In order to form edges corresponding to the edges of  $G$ , we traverse all the edges of  $G$  except the edges belonging to  $E$ .

**Definition 5.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called a *node addition* transformation if

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a partial, onto, and one-to-one relation and
2. Let the set of edges to be split be represented by an adjacency matrix  $E$ . Let  $\vec{E}_S = E \cdot \vec{1}$  be a vector denoting the source nodes of the edges in  $E$ . Let  $\vec{E}_T = \hat{E} \cdot \vec{1}$  be a vector denoting the target nodes of the edges in  $E$  and  $\vec{\eta} = \vec{1} - C \cdot \vec{1}$  be a vector denoting the new nodes. There exist a  $(|N| \times |N|)$  matrix  $E$ , a  $(|N| \times |N'|)$  matrix  $N_P$ , and a  $(|N'| \times |N|)$  matrix  $N_S$  such that the following conditions hold:
  - (a)  $E \leq A$ ,
  - (b)  $N_P$  is a total and onto relation from the nodes denoted by  $\vec{E}_S$  to the nodes denoted by  $\vec{\eta}$ ,
  - (c)  $N_S$  is a total and onto relation from the nodes denoted by  $\vec{\eta}$  to the nodes denoted by  $\vec{E}_T$ ,
  - (d)  $E = N_P \cdot N_S$ , and
  - (e)  $(C \cdot A \cdot \hat{C} - C \cdot E \cdot \hat{C}) + \underbrace{C \cdot N_P}_{E_I} + \underbrace{N_S \cdot \hat{C}}_{E_O} = A'$ .

The first condition ensures that there is at least one new node in the transformed graph (partiality) but no nodes of the input graph are deleted (ontoness). The one-to-one nature of the correspondence relation ensures that no nodes of the input graph are split or merged.

The second condition states the following: (a) The set  $E$  of the edges to be split is a subset of the edges of the input graph. (b)–(c) All the edges in  $E$  are split. (d) The edge splitting does not lead to any extraneous edges and any edges between the new nodes in the transformed graph. (e) It defines the adjacency matrix  $A'$  of the transformed graph in terms the adjacency matrix  $A$  of the input graph, the matrix  $E$ , and the relations  $C$ ,  $N_P$  and  $N_S$ .  $E_I$  and  $E_O$  respectively denote the incoming and outgoing edges for the new nodes as shown in Figure 5.

### 3.7. Node Deletion

A node deletion transformation of a graph  $G$  deletes at least one node. The incoming edges of the node being deleted are composed with its outgoing edges. The rest of the edges of  $G$  are preserved. For example, consider the two graphs shown in Figure 5 once again. This time consider  $G$  to be a transformation of  $G'$ . Node 4 of  $G'$  is deleted and edges  $\langle 3, 4 \rangle$  and  $\langle 4, 2 \rangle$  are joined to get edge  $\langle 3, 2 \rangle$  of  $G$ . The correspondence relation  $C$  is given as the dashed gray arrows from left to right (opposite to that of the node addition example). The

dotted gray arrows from left to right marked as  $N_P$  map the nodes in  $G'$  corresponding to the predecessors of the node being deleted to the node being deleted. The solid gray arrows from right to left marked as  $N_S$  map the node being deleted to the nodes in  $G'$  corresponding to the successors of the node being deleted. The correspondence between the edges of the two graphs can be traced diagrammatically.

**Definition 6.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called a *node deletion transformation* if all of the following conditions hold:

1. The correspondence relation  $C$  between the nodes of  $G'$  and  $G$  is a total and one-to-one but not onto.
2. Let  $\vec{\delta} = \vec{1} - \widehat{C} \cdot \vec{1}$  be a vector the nodes being deleted. Let  $\vec{\delta}_P = A \cdot \vec{\delta}$  be a vector denoting the predecessors of the nodes being deleted and  $\vec{\delta}_S = \widehat{A} \cdot \vec{\delta}$  be a vector denoting the successors of the nodes being deleted. We require that no two nodes having an edge between them can be deleted. We state this requirement as  $\vec{\delta} * (\vec{\delta}_P + \vec{\delta}_S) = \vec{0}$ .
3. Let  $\vec{\delta}_{P'} = C \cdot \vec{\delta}_P$  and  $\vec{\delta}_{S'} = C \cdot \vec{\delta}_S$  be the nodes in  $G'$  corresponding respectively to nodes denoted by  $\vec{\delta}_P$  and  $\vec{\delta}_S$ . Further, let  $E = (A \cdot A) * (\vec{\delta}_P \times \vec{\delta}_S)$  where  $\times$  is the Cartesian product operator. The set  $E$  gives the set of edges obtained by joining the incoming and outgoing edges of the nodes being deleted. There exist a  $(|N'| \times |N|)$  matrix  $N_P$ , and a  $(|N| \times |N'|)$  matrix  $N_S$  such that the following conditions hold:
  - (a)  $N_P$  is a total and onto relation from the nodes denoted by  $\vec{\delta}_{P'}$  to the nodes denoted by  $\vec{\delta}$ ,
  - (b)  $N_S$  is a total and onto relation from the nodes denoted by  $\vec{\delta}$  to the nodes denoted by  $\vec{\delta}_{S'}$ ,
  - (c)  $C \cdot E \cdot \widehat{C} = N_P \cdot N_S$ , and
  - (d)  $C \cdot A \cdot \widehat{C} + N_P \cdot N_S = A'$ .

The first condition states that there are no new nodes (totality) in the transformed graph and no nodes of the input graph are split or merged (one-to-one nature). However, at least one node of the input graph is deleted (the correspondence relation not being onto). The second condition states that no two nodes with an edge between them are deleted.

The third condition states the following: (a)–(b) The incoming edges of the nodes being deleted are joined with the outgoing edges. (c) The node deletion does not create any other edges (except those allowed by (a) and (b)) in the transformed graph. (d) It defines the adjacency matrix  $A'$  of the transformed graph in terms of the adjacency matrix  $A$  of the input graph and relations  $C$ ,  $N_P$ , and  $N_S$ .

### 3.8. Isomorphic Transformation

An isomorphic transformation transforms a graph into an isomorphic graph.

**Definition 7.** The transformation of a graph  $G = (N, A)$  to a graph  $G' = (N', A')$  is called an *isomorphic transformation* if

1. The correspondence  $C$  between the nodes of  $G'$  and  $G$  is a bijection and
2.  $C \cdot A \cdot \widehat{C} = A'$ .

## 4. Primitive program transformations

A primitive program transformation (or simply a transformation primitive) changes an input program in a small well-defined step. In this section, we present only the transformation primitives used in the CSE specification (Figure 1): insertion of predecessors (IP), insertion of assignments (IA), and replacement of expressions (RE). To define a primitive, we need to define the associated control flow and content transformations. A control flow transformation is modeled in terms of the primitive graph transformations (Section 3). This requires setting up various relations (matrices) used in the definitions of the primitive graph transformations.

Apart from the primitives presented here, we have also defined several other primitives viz. insertion of successors (IS), edge splitting (SE), deletion of statements (DS), and replacement of variable operands (RV). The formal definitions all the primitives are available in [Kan07]. These primitives have been used for defining most of the classical optimizations like common subexpression elimination, optimal code placement, loop invariant code motion, lazy code motion, and full and partial dead code elimination [Kan07].

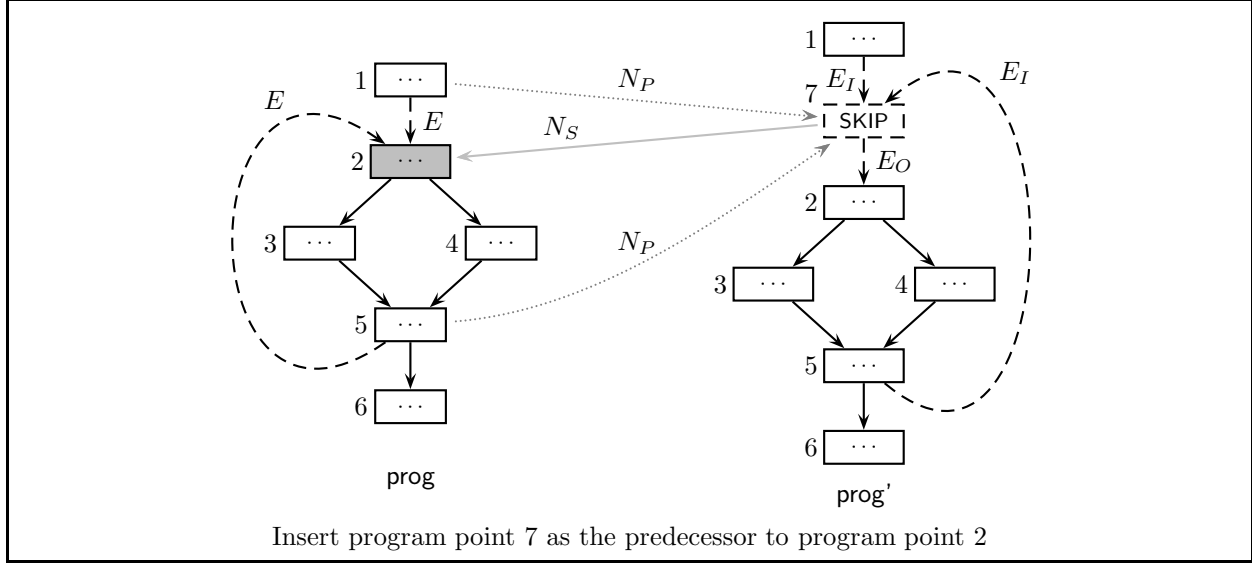


Fig. 6. An example of insertion of predecessors transformation

#### 4.1. Insertion of predecessors

Consider the two programs shown in Figure 6 such that  $\text{prog}' = \text{IP}(\text{prog}, \text{succs}, \text{newpoints})$ . The program  $\text{prog}'$  is obtained by inserting the new program point 7 as the predecessor to program point 2. Let us use the ordered sequence  $\langle 1, \dots, 6 \rangle$  for indexing vectors/matrices associated with  $\text{prog}$ . Let  $\text{succs} = \langle 0, 1, 0, 0, 0, 0 \rangle$  represent a set containing program point 2. Let  $\text{newpoints} = \langle 0, 0, 0, 0, 0, 0, 1 \rangle$  denote the set of new program points to be inserted as predecessors to  $\text{succs}$ . The ordered sequence for indexing vectors/matrices for  $\text{prog}'$  is  $\langle 1, \dots, 6, 7 \rangle$ . The new program point 7 is placed at the end of the list.

We model a transformation of the control flow graph of a program by an application of IP as a node addition transformation (Definition 5). Given the arguments of IP, we set up the adjacency matrices for the relations  $C$ ,  $N_S$ , and  $N_P$ . For the transformation in Figure 6, we have the following:

The relation  $C$  is represented as the matrix shown here. The rows correspond to program points  $1, \dots, 7$  (of the transformed program) and the columns correspond to program points  $1, \dots, 6$  (of the input program). Note that since the new program point does not correspond to any program point in the input graph, the last row (corresponding to program point 7) has all (boolean) 0s.

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	0

The matrix  $\text{Succs}$  is a  $(1 \times 6)$  matrix which is appended to a  $(6 \times 6)$  matrix containing all 0s to get the  $(7 \times 6)$  matrix  $N_S$  which maps program point 7 of  $\text{prog}'$  to program point 2 of  $\text{prog}$ .

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	1	0	0	0	0	0

The relation  $N_P$  (denoted as a matrix) maps program points 1 and 5 (the predecessors of program point 2 in  $\text{prog}$ ) to program point 7. Given the vector  $\text{succs}$ , we can identify the adjacency matrix  $E$  of the incoming edges to the program points denoted by  $\text{succs}$ . The matrix  $N_P$  is then obtained as  $E \cdot \widehat{N_S}$ .

0	0	0	0	0	0	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	0	0

It can be verified that the matrices satisfy the conditions about the nature of the corresponding relations stated in Definition 5. For example, the correspondence matrix  $C$  denotes a partial (at least one row has all 0s), onto (each column has at least one non-zero element), and one-to-one relation (each column as well as each row has at most one non-zero element). Clearly, the adjacency matrix  $A'$  of the control flow graph of  $\text{prog}'$  can be obtained by substituting these matrices and the adjacency matrix  $A$  of the control flow graph of  $\text{prog}$  in Definition 5.

An insertion of predecessors transformation inserts SKIP statements at the newly inserted program points. If the target of a goto or a conditional statement belongs to the set represented by `succs` then the target is updated to its new predecessor program point (identified using the  $N_S$  relation). For the example shown in Figure 6, the target of the conditional statement at program point 5 will be updated to program point 7 in the transformed program. All other statements remain unchanged.

## 4.2. Insertion of assignments

The insertion of assignments transformation primitive (IA) takes a program, a set of program points, a variable, and an expression as its arguments. An application  $IA(\text{prog}, \text{points}, v, e)$  of IA to a program `prog` inserts an assignment  $v := e$  at the program points denoted by the vector `points`. The rest of the statements remain unchanged. The control flow graph of the input program also remains unchanged and is defined as an isomorphic graph transformation (Definition 7).

## 4.3. Replacement of expressions

The replacement of expressions transformation primitive (RE) takes a program, a set of program points, an expression, and a variable as its arguments. An application  $RE(\text{prog}, \text{points}, e, v)$  of RE to a program `prog` replaces the occurrences of the expression `e` at the program points denoted by the vector `points`. The rest of the statements remain unchanged. The control flow graph of the input program also remains unchanged and is defined as an isomorphic graph transformation (Definition 7).

# 5. Program transformations as transformations of Kripke structures

Kripke structures serve as a natural modeling paradigm when the properties of interest are temporal in nature. In order to interpret temporal formulae, we abstract a program as a Kripke structure. The control flow graph of the program gives the transition relation of the Kripke structure. The atomic propositions of the Kripke structure correspond to the local data flow properties and the labeling function corresponds to the valuations of the local properties.

In this section, we model program transformations as transformations of Kripke structures. A program transformation may consist of a structural transformation of the control flow graph and a content transformation of the control flow graph nodes (statements). Similar to the program transformations, a structural transformation of a Kripke structure is defined using the primitive graph transformations. A content transformation is modeled by specifying modifications of atomic propositions and their labeling.

## 5.1. Transformations of Kripke structures

**Definition 8.** A *Kripke structure*  $M$  is a tuple  $(G, P, L)$  where  $G$  is a directed graph,  $P$  is a set of atomic propositions, and  $L : P \rightarrow \mathcal{B}_n$  is a labeling function that maps each atomic proposition in  $P$  to a boolean vector of size  $n$ . The  $i$ th element of the vector  $L(p)$  is (boolean) 1 iff the proposition  $p$  holds at the  $i$ th node.  $\mathcal{B}_n$  denotes the set of boolean vectors of size  $n$  where  $n$  is the number of nodes in  $G$ .

Following the usual assumption in program analysis but without loss of generality, we assume that  $G$  has a single entry and a single exit. Let  $A$  be the adjacency matrix of  $G$ . We consider an implicit ordering of the states of  $G$  for indexing vectors/matrices associated with  $M$ . We do not explicitly represent the states of a Kripke structure. We require  $A$  to be a total relation i.e. every state should have an outgoing edge. For an atomic proposition  $p$ , we use  $\vec{p}$  to denote  $L(p)$ . There are two special atomic propositions  $\varepsilon$  and  $\omega$  such that  $\vec{\varepsilon}$  and  $\vec{\omega}$  respectively hold only at the entry and the exit states.

**Definition 9.** A *Kripke transformation* maps a Kripke structure  $M = (G, P, L)$  to a Kripke structure  $M' = (G', P', L')$  and is defined as follows:

- (1) A graph transformation that maps  $G$  to  $G'$ . Let  $C$  as the correspondence matrix of the transformation.

- (2) For each atomic proposition  $p' \in P'$ , the labeling  $L'(p')$  (denoted as  $\vec{p}'$ ) is defined in terms of the labeling of an atomic proposition  $p \in P$  and other atomic propositions of  $M$  and  $M'$  as follows:

$$(C \cdot \vec{p} + \vec{u}') * \overline{\vec{v}'} * \vec{w}' = \vec{p}' \text{ where } u', v', \text{ and } w' \text{ are temporal formulae} \quad (1)$$

- (a)  $C \cdot \vec{p}$  indicates the states in  $M'$  which correspond to the states labeled by  $p$  in  $M$ ,
- (b)  $\vec{u}'$  indicates the states in  $M'$  which may not correspond to the states labeled with  $p$  in  $M$  but are labeled with  $p'$  in  $M'$ ,
- (c)  $\vec{v}'$  indicates the states in  $M'$  which are not labeled with  $p'$ , and
- (d)  $\vec{w}'$  indicates the states in  $M'$  which are labeled with  $p'$  and also with  $w'$ .

In general, the formulae  $u'$ ,  $v'$ , and  $w'$  in Equation (1) can be arbitrary CTL<sub>bp</sub> formulae possibly involving atomic propositions from both the input and the transformed Kripke structures. If there are  $m$  atomic propositions in  $M'$  then we get a system of  $m$  simultaneous equations. If the system is h-monotonic [KKS05] or simply monotonic then a solution to the equations can be computed iteratively and it completely determines the labeling of the transformed Kripke structure.

In our framework for verification of optimizations, we arrive at formulations of Kripke transformations only indirectly by modeling primitive program transformations as primitive Kripke transformations. Therefore, we do not solve formulations of Kripke transformations explicitly. Further, definitions of atomic propositions of the transformed Kripke structure involve only simple CTL<sub>bp</sub> formulae.

We identify a Kripke transformation with the type of the graph transformation involved. Thus we have node splitting, node merging, edge addition, edge deletion, node addition, node deletion, and isomorphic Kripke transformations whose component graph transformations are as defined in Section 3.

## 5.2. Insertion of predecessors as node addition Kripke transformation

In Section 2, we defined some local data flow properties viz. **Antloc**, **Transp**, and **Comp**. We now introduce a few more properties. A variable  $v$  is **Def** at a program point if  $v$  is assigned at the program point. A variable  $v$  is **Use** at a program point if  $v$  appears in an expression at the program point. A variable  $v$  and an expression  $e$  satisfy **AssignStmt** property at a program point if the statement at the program point is  $v := e$ .

Let  $\text{prog}_1 = \text{IP}(\text{prog}, \text{succs}, \text{newpoints})$  be an application of the insertion of predecessors primitive. Since an insertion of predecessors transformation involves a node addition graph transformation, we can abstract it as a node addition Kripke transformation (part 1 of Definition 9).

We now define the local data flow properties (atomic propositions) of the transformed program  $\text{prog}_1$  in terms of the local data flow properties of the input program  $\text{prog}$  (part 2 of Definition 9). Recall that IP inserts **SKIP** statements at the new program points and does not change statements at any other program point. Let  $C$  be the correspondence matrix for the (graph) transformation.

For any expression  $e$  and a variable  $v$  occurring in  $\text{prog}$  (and also in  $\text{prog}_1$ ), the correlation of the atomic propositions for the transformation is defined as follows:

$$\begin{aligned} C \cdot \text{Antloc}(\text{prog}, e) &= \text{Antloc}(\text{prog}_1, e) \\ C \cdot \text{Transp}(\text{prog}, e) + \text{newpoints} &= \text{Transp}(\text{prog}_1, e) \\ C \cdot \text{Comp}(\text{prog}, e) &= \text{Comp}(\text{prog}_1, e) \\ C \cdot \text{Mod}(\text{prog}, e) &= \text{Mod}(\text{prog}_1, e) \\ C \cdot \text{Use}(\text{prog}, v) &= \text{Use}(\text{prog}_1, v) \\ C \cdot \text{Def}(\text{prog}, v) &= \text{Def}(\text{prog}_1, v) \\ C \cdot \text{AssignStmt}(\text{prog}, v, e) &= \text{AssignStmt}(\text{prog}_1, v, e) \end{aligned}$$

Note that since the new program points (denoted by  $\text{newpoints}$ ) contain **SKIP** statement in  $\text{prog}_1$ , for any expression  $e$ ,  $\text{Transp}(\text{prog}_1, e)$  holds at  $\text{newpoints}$ .

Let  $\text{succs}' = C \cdot \text{succs}$  be a new atomic proposition denoting the program points in  $\text{prog}_1$  which correspond to the  $\text{succs}$  program points in  $\text{prog}$ . For each program point  $i$  from  $\text{succs}'$ , there exists a program point  $j$

from `newpoints` such that  $i$  is the only predecessor of  $j$  and  $j$  is the only successor of  $i$ . Hence,

$$\begin{aligned} \text{AX}(\text{prog}_1 \text{'cfg, succs'}) &= \text{newpoints} \\ \text{AY}(\text{prog}_1 \text{'cfg, newpoints'}) &= \text{succs'} \end{aligned}$$

Further, for each primitive program transformation, we lemmatize several other properties of the transformation or the transformed program. For instance, we introduce a lemma to state that the statements at the newly inserted program points `newpoints` are SKIP statements. The correctness of such lemmas can be proved from the definitions of the primitives.

### 5.3. Insertion of assignments as isomorphic transformation

Let  $\text{prog}_1 = \text{IA}(\text{prog}, \text{points}, \text{v}, \text{e})$  be an application of the insertion of assignments primitive. Since an insertion of assignments transformation does not change the control flow graph of the input program, we can abstract it as an isomorphic Kripke transformation (part 1 of Definition 9).

Recall that IA inserts an assignment  $\text{v} := \text{e}$  at the program points denoted by `points` and does not change statements at other program points. We assume that the statement at any program point in `points` in `prog` is SKIP. This condition is defined as a part of the soundness conditions for IA.

Below we correlate the local data flow properties of the input program `prog` and the transformed program  $\text{prog}_1$  (part 2 of Definition 9). `e1` is an expression and `v1` is a variable in the following:

$$\begin{aligned} \text{Antloc}(\text{prog}, \text{e1}) + \text{points} &= \text{Antloc}(\text{prog}_1, \text{e1}) && \text{if } \text{e} = \text{e1} \\ \text{Antloc}(\text{prog}, \text{e1}) &= \text{Antloc}(\text{prog}_1, \text{e1}) && \text{otherwise} \\ \text{Transp}(\text{prog}, \text{e1}) - \text{points} &= \text{Transp}(\text{prog}_1, \text{e1}) && \text{if } \text{v} \text{ is an operand of } \text{e1} \\ \text{Transp}(\text{prog}, \text{e1}) &= \text{Transp}(\text{prog}_1, \text{e1}) && \text{otherwise} \\ \text{Comp}(\text{prog}, \text{e1}) - \text{points} &= \text{Comp}(\text{prog}_1, \text{e1}) && \text{if } \text{v} \text{ is an operand of } \text{e1} \\ \text{Comp}(\text{prog}, \text{e1}) + \text{points} &= \text{Comp}(\text{prog}_1, \text{e1}) && \text{else if } \text{e} = \text{e1} \\ \text{Comp}(\text{prog}, \text{e1}) &= \text{Comp}(\text{prog}_1, \text{e1}) && \text{otherwise} \\ \text{Mod}(\text{prog}, \text{e1}) + \text{points} &= \text{Mod}(\text{prog}_1, \text{e1}) && \text{if } \text{v} \text{ is an operand of } \text{e1} \\ \text{Mod}(\text{prog}, \text{e1}) &= \text{Mod}(\text{prog}_1, \text{e1}) && \text{otherwise} \\ \text{Use}(\text{prog}, \text{v1}) + \text{points} &= \text{Use}(\text{prog}_1, \text{v1}) && \text{if } \text{v} \text{ is an operand of } \text{e1} \\ \text{Use}(\text{prog}, \text{v1}) &= \text{Use}(\text{prog}_1, \text{v1}) && \text{otherwise} \\ \text{Def}(\text{prog}, \text{v1}) + \text{points} &= \text{Def}(\text{prog}_1, \text{v1}) && \text{if } \text{v} = \text{v1} \\ \text{Def}(\text{prog}, \text{v1}) &= \text{Def}(\text{prog}_1, \text{v1}) && \text{otherwise} \\ \\ \text{AssignStmt}(\text{prog}, \text{v1}, \text{e1}) + \text{points} &= \text{AssignStmt}(\text{prog}_1, \text{v1}, \text{e1}) && \text{if } \text{v} = \text{v1} \text{ and } \text{e} = \text{e1} \\ \text{AssignStmt}(\text{prog}, \text{v1}, \text{e1}) &= \text{AssignStmt}(\text{prog}_1, \text{v1}, \text{e1}) && \text{otherwise} \end{aligned}$$

### 5.4. Replacement of expressions as isomorphic transformation

Let  $\text{prog}_1 = \text{RE}(\text{prog}, \text{points}, \text{e}, \text{v})$  be an application of the replacement of expressions primitive. Since a replacement of expressions transformation does not change the control flow graph of the input program, we can abstract it as an isomorphic Kripke transformation (part 1 of Definition 9).

Recall that RE replaces the occurrences of expression `e` at the program points denoted by `points` by variable `v` and does not change statements at other program points.

Below we correlate local data flow properties of the input program `prog` and the transformed program

$\text{prog}_1$  (part 2 of Definition 9).  $e1$  is an expression and  $v1$  is a variable in the following:

$$\begin{aligned}
\text{Antloc}(\text{prog}, e1) - \text{points} &= \text{Antloc}(\text{prog}_1, e1) && \text{if } e = e1 \\
\text{Antloc}(\text{prog}, e1) + \text{points} &= \text{Antloc}(\text{prog}_1, e1) && \text{if the expression } e1 \text{ is just the variable } v \\
\text{Antloc}(\text{prog}, e1) &= \text{Antloc}(\text{prog}_1, e1) && \text{otherwise} \\
\text{Transp}(\text{prog}, e1) &= \text{Transp}(\text{prog}_1, e1) \\
\text{Comp}(\text{prog}, e1) - \text{points} &= \text{Comp}(\text{prog}_1, e1) && \text{if } e = e1 \\
\text{Comp}(\text{prog}, e1) &\leq \text{Comp}(\text{prog}_1, e1) && \text{if the expression } e1 \text{ is just the variable } v \\
\text{Comp}(\text{prog}, e1) &= \text{Comp}(\text{prog}_1, e1) && \text{otherwise} \\
\text{Mod}(\text{prog}, e1) &= \text{Mod}(\text{prog}_1, e1) \\
\text{Use}(\text{prog}, v1) - \text{points} &= \text{Use}(\text{prog}_1, v1) && \text{if } v1 \text{ is an operand of } e \\
\text{Use}(\text{prog}, v1) + \text{points} &= \text{Use}(\text{prog}_1, v1) && \text{else if } v = v1 \\
\text{Use}(\text{prog}, v1) &= \text{Use}(\text{prog}_1, v1) && \text{otherwise} \\
\text{Def}(\text{prog}, v1) &= \text{Def}(\text{prog}_1, v1)
\end{aligned}$$

Note the inequality in the second correlation of the **Comp** property. Suppose the expression  $e1 \neq e$  is just the variable  $v$ . If  $e1$  is **Comp** at a program point  $i$  in the input program then it is also **Comp** at  $i$  in the transformed program. Therefore,  $\text{Comp}(\text{prog}, e1) \leq \text{Comp}(\text{prog}_1, e1)$ . At the program points denoted by **points**, the expression  $e1$  is **Antloc** in the transformed program. However, depending on whether the left-hand side variable at a program point is  $v$  or not, it may or may not be **Comp** at that point. Since we do not require a complete characterization of  $\text{Comp}(\text{prog}_1, e1)$  in the proofs, we prefer the inequality as the correlation between **Comp** program points. For similar reasons, we prefer an inequality in the second correlation of **AssignStmt** property:

$$\begin{aligned}
\text{AssignStmt}(\text{prog}, v1, e1) - \text{points} &= \text{AssignStmt}(\text{prog}', v1, e1) && \text{if } e = e1 \\
\text{AssignStmt}(\text{prog}, v1, e1) &\leq \text{AssignStmt}(\text{prog}', v1, e1) && \text{if the expression } e1 \text{ is just the variable } v1 \\
\text{AssignStmt}(\text{prog}, v1, e1) &= \text{AssignStmt}(\text{prog}', v1, e1) && \text{otherwise}
\end{aligned}$$

## 6. Temporal transformation logic

In this section, we introduce the logic TTL. We present the syntax and the semantics of TTL operators. We then define the axioms and the basic inference rules. For each type of primitive Kripke transformations, we present the inference rules to correlate temporal properties between a Kripke structure and its transformation under the specific primitive transformation.

### 6.1. Syntax and semantics

We give boolean matrix algebraic semantics to  $\text{CTL}_{\text{bp}}$  formulae. For a formula  $\varphi$ , we denote the set of states where the the formula is satisfied by a boolean vector  $\vec{\varphi}$ . For an atomic proposition  $p$ , the labeling  $L(p)$  is denoted by a boolean vector  $\vec{p}$ . The one-step temporal modalities can be defined by matrix multiplication. For example, the semantics of the formula  $\text{EX}(\varphi)$  is given by  $A \cdot \vec{\varphi}$  where  $A$  is the boolean adjacency matrix of the transition relation of the Kripke structure and the operator “ $\cdot$ ” denotes matrix multiplication. The semantics of path operators (until, globally, etc.) are specified by fixed points. Appendix A gives the syntax and semantics of  $\text{CTL}_{\text{bp}}$  formulae.

Consider a Kripke structure  $M$  and its transformed version  $M'$ . Let  $\varphi$  and  $\varphi'$  be  $\text{CTL}_{\text{bp}}$  formulae defined respectively over Kripke structures  $M$  and  $M'$ . As a convention, we use primed symbols for a transformed Kripke structure and unprimed symbols for an input Kripke structure. Correlations between  $\text{CTL}_{\text{bp}}$  formulae  $\varphi$  and  $\varphi'$  are denoted the following TTL formulae:

$$\varphi \rightarrow \varphi', \varphi \Rightarrow \varphi', \text{ and } \varphi \leftarrow \varphi' \tag{2}$$



The model of a TTL formula is the pair  $\langle M, M' \rangle$  of Kripke structures. Let  $C$  be the correspondence matrix that relates the nodes of  $M'$  and  $M$ . The semantics of the TTL formulae are defined as follows:

$$\begin{aligned} \langle M, M' \rangle \models \varphi \rightarrow \varphi' & \text{ iff } C \cdot \vec{\varphi} \leq \vec{\varphi}' \\ \langle M, M' \rangle \models \varphi \Rightarrow \varphi' & \text{ iff } C \cdot \vec{\varphi} = \vec{\varphi}' \\ \langle M, M' \rangle \models \varphi \leftarrow \varphi' & \text{ iff } C \cdot \vec{\varphi} \geq \vec{\varphi}' \end{aligned} \quad (3)$$

## 6.2. Axioms and basic inference rules

An inference rule is represented as follows:

$$\frac{\alpha_1 \quad \cdots \quad \alpha_k}{\beta}$$

(name of the rule and its scope)

where the judgments  $\alpha_1, \dots, \alpha_k$  above the line denote the premises and the judgment below the line denotes the conclusion. A premise  $\alpha_i$  can be a  $\text{CTL}_{\text{bp}}$  formula defined either on an input Kripke structure or on its transformation, or it can be a TTL formula. The conclusion  $\beta$  is a TTL formula. An inference rule has a name. The scope of the rule specifies the type of Kripke transformations for which the inference is valid. We abbreviate the transformation primitives by their first letters e.g. node splitting is abbreviated as NS. If no specific scope is given then the rule is applicable to all primitive transformations.

Consider a pair  $\langle M, M' \rangle$  of Kripke structures where  $M'$  is a transformation of  $M$  according to some transformation within the scope of an inference rule. The inference rule states that if the premises hold for  $\langle M, M' \rangle$  then the conclusion is also holds for  $\langle M, M' \rangle$ .

In Figure 7, we give the axioms and basic inference rules of the logic. The correspondence matrix for a pair of Kripke structures  $\langle M, M' \rangle$  is denoted by  $C$ . The axiom AXM1 is applicable if an atomic proposition  $p$  defined on  $M$  and an atomic proposition  $p'$  defined on  $M'$  satisfy the condition  $C \cdot \vec{p} \leq \vec{p}'$ . In that case,  $p \rightarrow p'$  follows directly. Note that the condition  $C \cdot \vec{p} \leq \vec{p}'$  is not part of the logic. The axiom allows introduction of atomic judgments (judgments involving atomic propositions only) in a proof. Similarly, AXM2 and AXM3.

Proofs of the basic inference rules are straightforward. We only discuss a few interesting ones here. The rule MP2 is applicable to node addition transformations only. The atomic proposition  $\eta'$  denotes the set of new nodes in the transformed Kripke structure  $M'$ . It is defined as  $\vec{\eta}' = \vec{1} - C \cdot \vec{1}$ . MP2 is sound because (1)  $\vec{1} \leq \vec{\varphi}$  : given, (2)  $C \cdot \vec{1} \leq C \cdot \vec{\varphi}$  : monotonic, (3)  $C \cdot \vec{\varphi} \leq \vec{\varphi}'$  : given, (4)  $C \cdot \vec{1} \leq \vec{\varphi}'$  : transitive, (5)  $C \cdot \vec{1} + (\vec{1} - C \cdot \vec{1}) \leq \vec{\varphi}' + \vec{\eta}'$  : monotonic and definition of  $\eta'$ , and (6)  $\vec{1} \leq \vec{\varphi}' + \vec{\eta}'$ .

The rule MP3 is applicable to node deletion transformations only. The atomic proposition  $\delta$  denotes the set of nodes that are deleted from the input Kripke structure. MP3 is sound because (1)  $\vec{1} \leq \vec{\varphi} + \vec{\delta}$  : given, (2)  $\vec{1} - \vec{\delta} \leq \vec{\varphi}$ , (3)  $C \cdot (\vec{1} - \vec{\delta}) \leq C \cdot \vec{\varphi}$  : monotonic, (4)  $C \cdot \vec{\varphi} \leq \vec{\varphi}'$  : given, (5)  $C \cdot (\vec{1} - \vec{\delta}) = \vec{1}$  :  $C$  is a total but not onto function and does not relate the nodes of  $M'$  with the nodes being deleted, and (6)  $\vec{1} \leq \vec{\varphi}'$  : transitive.

The rule NI1 is applicable to the primitives which satisfy the constraint  $C \cdot \vec{z} \geq \overline{C \cdot \vec{z}}$  for any  $\vec{z} \in \mathcal{B}_n$  where  $n$  is the number of nodes of the input Kripke structure. NI1 is sound because (1)  $C \cdot \vec{\varphi} \leq \vec{\varphi}'$  : given, (2)  $\overline{C \cdot \vec{\varphi}} \geq \overline{\vec{\varphi}'}$  : negation, (3)  $C \cdot \vec{\varphi} \geq \overline{C \cdot \vec{\varphi}}$  : since  $C \cdot \vec{z} \geq \overline{C \cdot \vec{z}}$ , and (4)  $C \cdot \vec{\varphi} \geq \overline{\vec{\varphi}'}$  : transitive. The rule NI2 is applicable to the primitives which satisfy the constraint that  $C \cdot \vec{z} \leq \overline{C \cdot \vec{z}}$ . NI2 is sound because (1)  $C \cdot \vec{\varphi} = \vec{\varphi}'$  : given, (2)  $\overline{C \cdot \vec{\varphi}} = \overline{\vec{\varphi}'}$  : negation, (3)  $C \cdot \vec{\varphi} \leq \overline{C \cdot \vec{\varphi}}$  : since  $C \cdot \vec{z} \leq \overline{C \cdot \vec{z}}$ , and (4)  $C \cdot \vec{\varphi} \leq \overline{\vec{\varphi}'}$  : transitive.

The relation  $C \cdot \vec{z} \geq \overline{C \cdot \vec{z}}$  means that for any  $\vec{z} \in \mathcal{B}_n$ , the set of nodes of  $M'$  corresponding to the complement of  $\vec{z}$  is a superset of the set of nodes of  $M'$  obtained by complement of the set of nodes (of  $M'$ ) corresponding to  $\vec{z}$ . Conversely,  $C \cdot \vec{z} \leq \overline{C \cdot \vec{z}}$  means that the former set is a subset of the later set. We now establish the relations between  $C \cdot \vec{z}$  and  $\overline{C \cdot \vec{z}}$  for the primitive transformations.

**Lemma 10.** If  $C$  is the correspondence matrix of an NS, EA, ED, or ND transformation from  $M$  to  $M'$  and  $\vec{z}$  is a vector on  $M$  then  $C \cdot \vec{z} = \overline{C \cdot \vec{z}}$ .

$\frac{}{p \rightarrow p'}$ (AXM1 if $C \cdot \vec{p} \leq \vec{p}'$ )	$\frac{}{p \Rightarrow p'}$ (AXM2 if $C \cdot \vec{p} = \vec{p}'$ )	$\frac{}{p \leftarrow p'}$ (AXM3 if $C \cdot \vec{p} \geq \vec{p}'$ )
<b>(a) Axioms for introducing atomic judgments</b>		
$\frac{\varphi \supset \psi \quad \psi \rightarrow \psi'}{\varphi \rightarrow \psi'}$ (FC1)	$\frac{\varphi \rightarrow \varphi' \quad \varphi' \supset \psi'}{\varphi \rightarrow \psi'}$ (FC2)	
<b>(b) Forward Chaining</b>		
$\frac{\top \supset \varphi \quad \varphi \rightarrow \varphi'}{\top \supset \varphi'}$ (MP1 for NS, NM, EA, ED, IM)	$\frac{\top \supset \varphi \quad \varphi \rightarrow \varphi'}{\top \supset (\varphi' \vee \eta')}$ (MP2 for NA)	$\frac{\top \supset (\varphi \vee \delta) \quad \varphi \rightarrow \varphi'}{\top \supset \varphi'}$ (MP3 for ND)
<b>(c) Modus Ponens for TTL</b>		
$\frac{\varphi \rightarrow \varphi' \quad \varphi' \supset \perp'}{\varphi \supset \perp}$ (MT1 for NS, NM, EA, ED, IM)	$\frac{\varphi \rightarrow \varphi' \quad (\varphi' \wedge \neg \eta') \supset \perp'}{\varphi \supset \perp}$ (MT2 for NA)	$\frac{\varphi \rightarrow \varphi' \quad \varphi' \supset \perp'}{(\varphi \wedge \neg \delta) \supset \perp}$ (MT3 for ND)
<b>(d) Modus Tollens for TTL</b>		
$\frac{\varphi \rightarrow \varphi' \quad \psi \rightarrow \varphi'}{(\varphi \vee \psi) \rightarrow \varphi'}$ (DI1)	$\frac{\varphi \rightarrow \varphi'}{\varphi \rightarrow (\varphi' \vee \psi')}$ (DI2)	$\frac{\varphi \rightarrow \varphi'}{(\varphi \wedge \psi) \rightarrow \varphi'}$ (CI1)
<b>(e) Disjunct Introduction</b>		$\frac{\varphi \rightarrow \varphi' \quad \varphi \rightarrow \psi'}{\varphi \rightarrow (\varphi' \wedge \psi')}$ (CI2)
<b>(f) Conjunct Introduction</b>		
$\frac{\varphi \rightarrow \varphi'}{\neg \varphi \leftarrow \neg \varphi'}$ (NI1 if $C \cdot \vec{z} \geq \overline{C \cdot \vec{z}}$ )	$\frac{\varphi \Rightarrow \varphi'}{\neg \varphi \rightarrow \neg \varphi'}$ (NI2 if $C \cdot \vec{z} \leq \overline{C \cdot \vec{z}}$ )	
<b>(g) Negation Introduction</b>		
$\frac{\varphi \Rightarrow \varphi'}{\varphi \rightarrow \varphi'}$ (IR1)	$\frac{\varphi \Rightarrow \varphi'}{\varphi \leftarrow \varphi'}$ (IR2)	$\frac{\varphi \supset \psi \quad \varphi \Rightarrow \varphi' \quad \psi \Rightarrow \psi'}{\varphi' \supset \psi'}$ (IT)
<b>(h) Implication Relaxation</b>		<b>(i) Implication Transfer</b>
$\frac{\varphi \leftarrow \varphi' \quad \varphi \rightarrow \psi'}{\varphi' \supset \psi'}$ (MC)	$\frac{\varphi \leftarrow \varphi' \quad \varphi' \subset \psi'}{\varphi \leftarrow \psi'}$ (BC1)	$\frac{\varphi \subset \psi \quad \psi \leftarrow \psi'}{\varphi \leftarrow \psi'}$ (BC2)
<b>(j) Mixed Chaining</b>		<b>(k) Backward Chaining</b>
$\frac{(\varphi \vee \psi) \rightarrow \varphi'}{\varphi \rightarrow \varphi'}$ (DE)	$\frac{\varphi \rightarrow (\varphi' \wedge \psi')}{\varphi \rightarrow \varphi'}$ (CE)	$\frac{\varphi' \supset \psi' \quad \varphi \Rightarrow \varphi' \quad \psi \Rightarrow \psi'}{\varphi \supset \psi}$ (IRT if $\vec{z} = \widehat{C} \cdot C \cdot \vec{z}$ )
<b>(l) Disjunct/Conjunct Elimination</b>		<b>(m) Implication Reverse Transfer</b>
$\frac{\neg \varphi \rightarrow \neg \varphi'}{\varphi \leftarrow \varphi'}$ (NE1 if $C \cdot \vec{z} \geq \overline{C \cdot \vec{z}}$ )	$\frac{\neg \varphi \leftarrow \neg \varphi'}{\varphi \rightarrow \varphi'}$ (NE2 if $C \cdot \vec{z} \leq \overline{C \cdot \vec{z}}$ )	
<b>(n) Negation Elimination</b>		

Fig. 7. Axioms and basic inference rules

*Proof.* Consider a boolean vector  $\vec{z}$  defined over  $M$ .

$$\begin{aligned}
 \vec{z} + \bar{\vec{z}} &= \vec{1} && \text{identity element} \\
 C \cdot (\vec{z} + \bar{\vec{z}}) &= C \cdot \vec{1} && \text{monotonic} \\
 (i) \quad C \cdot \vec{z} + C \cdot \bar{\vec{z}} &= \vec{1} && \text{distributive; } C \text{ is a total relation, hence } C \cdot \vec{1} = \vec{1}
 \end{aligned}$$

Consider the following derivation.

$$\begin{aligned}
 \vec{z} * \bar{\vec{z}} &= \vec{0} && \text{identity element} \\
 C \cdot (\vec{z} * \bar{\vec{z}}) &= C \cdot \vec{0} && \text{monotonic} \\
 (ii) \quad C \cdot \vec{z} * C \cdot \bar{\vec{z}} &= \vec{0} && \text{since the relation } C \text{ is a function, it distributes}
 \end{aligned}$$

We first show that  $C \cdot \bar{\vec{z}} \leq \overline{C \cdot \vec{z}}$ .

$$\begin{aligned}
 \vec{1} &= \vec{0} \\
 C \cdot \vec{z} + C \cdot \bar{\vec{z}} &= \overline{C \cdot \vec{z} * C \cdot \bar{\vec{z}}} && \text{from (i) and (ii) above} \\
 (iii) \quad (C \cdot \vec{z} + C \cdot \bar{\vec{z}}) * C \cdot \bar{\vec{z}} &= (\overline{C \cdot \vec{z} + C \cdot \bar{\vec{z}}}) * C \cdot \bar{\vec{z}} && \text{monotonic} \\
 (C \cdot \vec{z} * C \cdot \bar{\vec{z}}) + (C \cdot \bar{\vec{z}} * C \cdot \bar{\vec{z}}) &= (\overline{C \cdot \vec{z} * C \cdot \bar{\vec{z}}}) + (\overline{C \cdot \bar{\vec{z}} * C \cdot \bar{\vec{z}}}) && \text{distributive} \\
 \vec{0} + C \cdot \bar{\vec{z}} &= (\overline{C \cdot \vec{z} * C \cdot \bar{\vec{z}}}) + \vec{0} && \text{from (ii), identity element} \\
 C \cdot \bar{\vec{z}} &\leq \overline{C \cdot \vec{z}}
 \end{aligned}$$

Instead of  $C \cdot \bar{\vec{z}}$ , if we take a product by  $\overline{C \cdot \vec{z}}$  in (iii), we get  $C \cdot \bar{\vec{z}} \geq \overline{C \cdot \vec{z}}$ .  $\square$

**Lemma 11.** If  $C$  is the correspondence matrix of an NM transformation from  $M$  to  $M'$  and  $\vec{z}$  is a vector on  $M$  then  $C \cdot \bar{\vec{z}} \geq \overline{C \cdot \vec{z}}$ .

The proof of Lemma 11 is similar to that of Lemma 10. From Lemma 10 and Lemma 11, we know that the rule NI1 is applicable to the primitives NS, EA, ED, ND, and NM.

**Lemma 12.** If  $C$  is the correspondence matrix of an NA transformation from  $M$  to  $M'$ ,  $\vec{z}$  is a vector on  $M$ , and  $\eta'$  the atomic proposition denoting the new nodes in  $M'$  then  $\overline{C \cdot \vec{z}} * \eta' = C \cdot \vec{z}$ .

*Proof.* Consider a boolean vector  $\vec{z}$  defined over  $M$ .

$$\begin{aligned}
 \vec{z} + \bar{\vec{z}} &= \vec{1} \\
 C \cdot (\vec{z} + \bar{\vec{z}}) &= C \cdot \vec{1} && \text{monotonic} \\
 C \cdot \vec{z} + C \cdot \bar{\vec{z}} &= C \cdot \vec{1} \leq \vec{1} && \text{distribute; } C \text{ being a partial function, } C \cdot \vec{1} \leq \vec{1}
 \end{aligned}$$

Consider the following derivation.

$$\begin{aligned}
 \vec{z} * \bar{\vec{z}} &= \vec{0} \\
 C \cdot (\vec{z} * \bar{\vec{z}}) &= C \cdot \vec{0} && \text{monotonic} \\
 (i) \quad C \cdot \vec{z} * C \cdot \bar{\vec{z}} &= \vec{0} && \text{distribute; } C \text{ is a function}
 \end{aligned}$$

Since  $\eta' = \vec{1} - C \cdot \vec{1}$ , we have  $C \cdot \vec{z} + C \cdot \bar{\vec{z}} + \eta' = \vec{1}$ .

We first show that  $C \cdot \bar{z} \leq \overline{C \cdot \bar{z}} * \bar{\eta}'$ .

$$\begin{array}{llll}
\bar{1} & = & \bar{0} & \\
C \cdot \bar{z} + C \cdot \bar{z} + \bar{\eta}' & = & \overline{C \cdot \bar{z} * C \cdot \bar{z}} & \text{substitute} \\
(C \cdot \bar{z} + C \cdot \bar{z} + \bar{\eta}') * C \cdot \bar{z} & = & (\overline{C \cdot \bar{z}} + \overline{C \cdot \bar{z}}) * C \cdot \bar{z} & \text{distributive, monotonic} \\
(ii) \quad \bar{0} + C \cdot \bar{z} + (\bar{\eta}' * C \cdot \bar{z}) & = & (\overline{C \cdot \bar{z}} * C \cdot \bar{z}) + \bar{0} & \text{distributive, substitute (i)} \\
C \cdot \bar{z} & \leq & \overline{C \cdot \bar{z}} * C \cdot \bar{z} & \\
(iii) \quad C \cdot \bar{z} & \leq & \overline{C \cdot \bar{z}} & \\
\bar{\eta}' * C \cdot \bar{z} & \leq & \overline{C \cdot \bar{z}} * C \cdot \bar{z} & \text{third disjunct of LHS (ii)} \\
(iv) \quad \bar{\eta}' & \leq & \overline{C \cdot \bar{z}} & \\
C \cdot \bar{z} + \bar{\eta}' & \leq & \overline{C \cdot \bar{z}} & (iii), (iv), \text{monotonic, distributive} \\
C \cdot \bar{z} & \leq & \overline{C \cdot \bar{z}} * \bar{\eta}' & \text{negation; } C \cdot \bar{z} \text{ and } \bar{\eta}' \text{ are disjoint}
\end{array}$$

Instead of  $C \cdot \bar{z}$ , if we take a product by  $\overline{C \cdot \bar{z}} * \bar{\eta}'$  in the third step, we get  $C \cdot \bar{z} \geq \overline{C \cdot \bar{z}} * \bar{\eta}'$ .  $\square$

From Lemma 12, for the primitive NA,  $C \cdot \bar{z} \leq \overline{C \cdot \bar{z}}$ . Thus, additionally from Lemma 10, we know that the rule NI2 is applicable to the primitives NS, EA, ED, ND, and NA.

The rule IRT is applicable to the primitives which satisfy the constraint  $\bar{z} = \widehat{C} \cdot C \cdot \bar{z}$  for any  $\bar{z} \in \mathcal{B}_n$  where  $n$  is the number of nodes of the input Kripke structure. IRT is sound because (1)  $\bar{\varphi}' \leq \bar{\psi}'$  : given, (2)  $\widehat{C} \cdot \bar{\varphi}' \leq \widehat{C} \cdot \bar{\psi}'$  : monotonic, (3)  $C \cdot \bar{\varphi} = \bar{\varphi}'$  : given, (4)  $\widehat{C} \cdot C \cdot \bar{\varphi} = \widehat{C} \cdot \bar{\varphi}'$  : monotonic, (5)  $\bar{\varphi} = \widehat{C} \cdot \bar{\varphi}'$  : since  $\bar{z} = \widehat{C} \cdot C \cdot \bar{z}$ , (6)  $C \cdot \bar{\psi} = \bar{\psi}'$  : given, (7)  $\widehat{C} \cdot C \cdot \bar{\psi} = \widehat{C} \cdot \bar{\psi}'$  : monotonic, (8)  $\bar{\psi} = \widehat{C} \cdot \bar{\psi}'$  : since  $\bar{z} = \widehat{C} \cdot C \cdot \bar{z}$ , and (9)  $\bar{\varphi} \leq \bar{\psi}$  : substitute from (5) and (8) in (2).

From the nature of the correspondence relations of the primitives, we have the following result.

**Lemma 13.** If  $C$  is the correspondence matrix of a primitive transformation of  $M$  to  $M'$  and  $Z$  and  $Z'$  are boolean matrices of appropriate sizes then the following properties hold:

1. For an NS, EA, ED, NA, or IM transformation,  $Z = \widehat{C} \cdot C \cdot Z$  and  $Z = Z \cdot \widehat{C} \cdot C$ .
2. For an NM transformation,  $Z \leq \widehat{C} \cdot C \cdot Z$  and  $Z \leq Z \cdot \widehat{C} \cdot C$ .
3. For an ED, ND, or IM transformation,  $Z' = C \cdot \widehat{C} \cdot Z'$  and  $Z' = Z' \cdot C \cdot \widehat{C}$ .
4. For an NS transformation,  $Z' \leq C \cdot \widehat{C} \cdot Z'$  and  $Z' \leq Z' \cdot C \cdot \widehat{C}$ .

From Lemma 13, it is clear that  $\bar{z} = \widehat{C} \cdot C \cdot \bar{z}$  holds for NS, EA, ED, NA, and IM transformations and thus the rule IRT is applicable to them.

### 6.3. Transformation specific inference rules

The transformation specific rules define sound inferences for introduction of temporal operators in the TTL formulae. The rules we present in Figure 8 allow introduction of the same temporal operators on both sides of a TTL implication. The rules are different for different types of primitives transformations. We defer the proofs of soundness of the rules until Section 7. Here, we discuss them only informally. The boolean matrix algebraic semantics of  $\text{CTL}_{\text{bp}}$  operators is given in Appendix A.

An NM or an EA transformation inserts new edges in the transformed Kripke structure but preserves all the existing edges. Thus, we can correlate any existential temporal operator between the input and the transformed Kripke structures.

An ED transformation, on the other hand, deletes edges from the input Kripke structure but does not add any edges. Thus, we can correlate any universal temporal operator between the input and the transformed Kripke structures. However,  $\text{TTL}_{\text{ed}}$  rules are applicable only for future operators. In  $\text{CTL}_{\text{bp}}$ , past operator AY (Definition 27) is interpreted in a strong sense i.e. for an AY formula to hold at a node, it must have at least one predecessor. At an entry node i.e. a node without any predecessors, no AY formula holds. For edge deletion, we cannot guarantee existence of a predecessor for every node in the transformed Kripke structure.

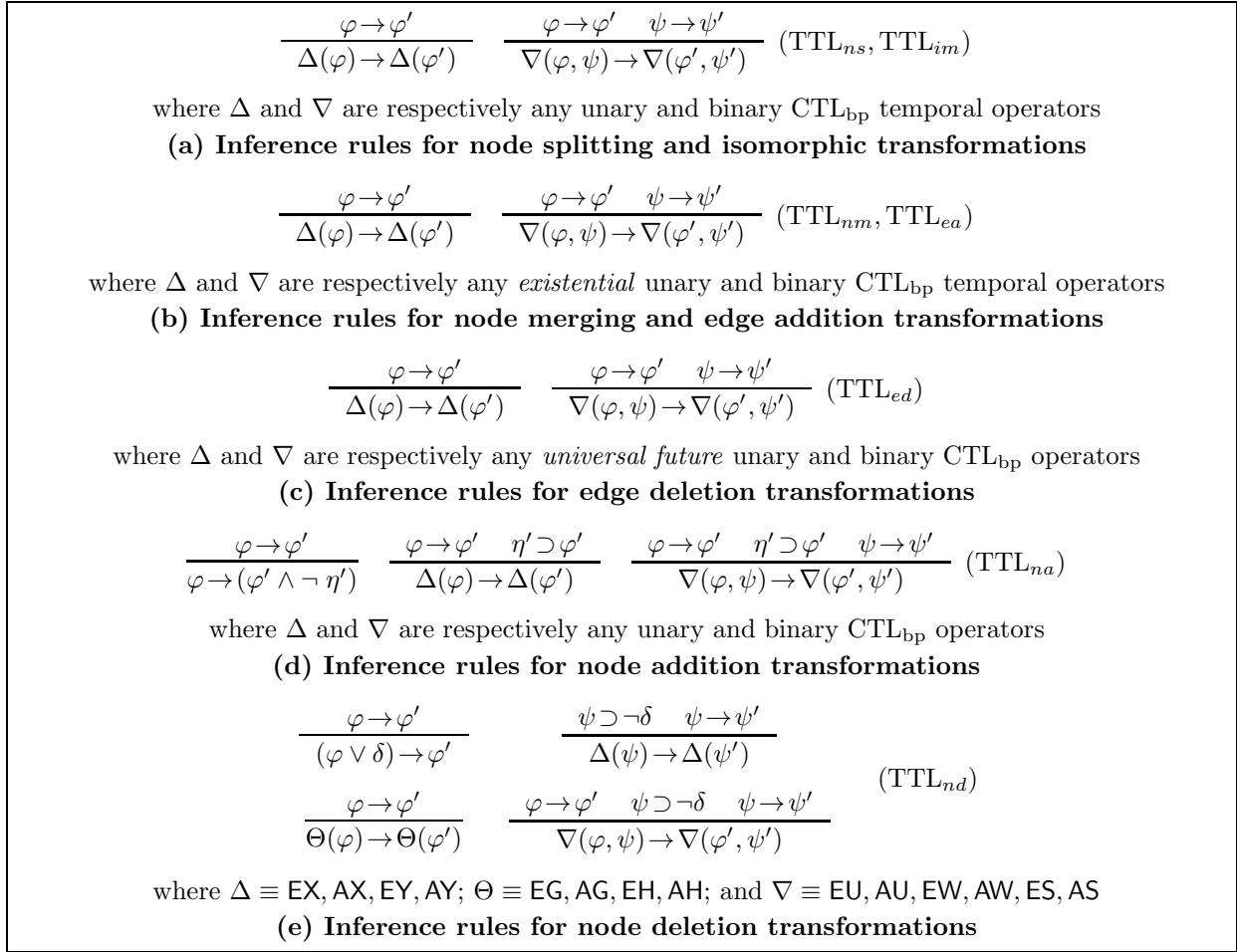


Fig. 8. Transformation specific inference rules

The existence of successors however is guaranteed because for a Kripke structure to be well-formed, every node has to have an outgoing edge. Hence, we can correlate universal future  $\text{CTL}_{bp}$  operators.

In Figure 8 (d),  $\eta'$  is an atomic proposition of a transformed Kripke structure that denotes the newly added nodes by an NA transformation. In Figure 8 (e),  $\delta$  is an atomic proposition of an input Kripke structure that denotes the nodes being deleted by an ND transformation.

#### 6.4. Verification of compiler optimizations revisited

In Section 2, we discussed a verification scheme for specifications of compiler optimizations that motivated the development of TTL. We now give a sketch of the proof of a verification condition for the CSE specification (Figure 1) using TTL. We have used TTL to prove soundness of several optimizations viz. common subexpression elimination, optimal code placement, loop invariant code motion, lazy code motion, and full and partial dead code elimination, in the PVS theorem prover.

Consider the last transformation  $\text{RE}(\text{prog4}, \text{redund4}, \text{e}, \text{t})$ . Common subexpression elimination is performed for non trivial expressions i.e. expressions containing some operator. Thus we need to establish part (b) of the constraint (4) in the soundness condition for the primitive RE. Formally, we need to show that

$$\text{redund4} \supset \text{AY}(\text{prog4}'\text{cfg}, \text{AS}(\text{prog4}'\text{cfg}, \text{Transp}(\text{prog4}, \text{e}) * \neg \text{Def}(\text{prog4}, \text{t}), \text{AssignStmt}(\text{prog4}, \text{t}, \text{e}))) \equiv \varphi_4 \quad (4)$$

From the definition of Avail, we know that

$$\text{redund} \supset \text{AY}(\text{prog1'cfg}, \text{AS}(\text{prog1'cfg}, \text{Transp}(\text{prog1,e}), \text{orgavails}))$$

We can prove that  $\text{orgavails} \supset \text{Transp}(\text{prog1,e})$ . Let  $\text{redund2}$  be the set of program points in  $\text{prog2}$  that correspond to  $\text{redund}$ . Since, IP is a node addition transformation, using  $\text{TTL}_{na}$ ; and the correlations of local data flow properties under an IP transformation (Section 5.2),

$$\text{redund2} \supset \text{AY}(\text{prog2'cfg}, \text{AS}(\text{prog2'cfg}, \text{Transp}(\text{prog2,e}), \text{newpoints}))$$

The variable  $t$  is declared as a new variable for  $\text{prog2}$ . Thus,  $\neg\text{Def}(\text{prog2},t)$  is an invariant property. Hence,

$$\text{redund2} \supset \text{AY}(\text{prog2'cfg}, \text{AS}(\text{prog2'cfg}, \text{Transp}(\text{prog2,e}) * \neg\text{Def}(\text{prog2},t), \text{newpoints}))$$

Since, IA is an isomorphic transformation, using  $\text{TTL}_{im}$ ; and the correlations of local data flow properties under an IA transformation (Section 5.3),

$$\text{redund2} \supset \text{AY}(\text{prog3'cfg}, \text{AS}(\text{prog3'cfg}, \text{Transp}(\text{prog3,e}) * \neg\text{Def}(\text{prog3},t), \text{AssignStmt}(\text{prog3},t,e))) \equiv \varphi_3 \quad (5)$$

Now, consider the following derivation:

$$\begin{array}{lll} \text{Def}(\text{prog3},t) & \Rightarrow & \text{Def}(\text{prog4},t) & \text{Section 5.4} \\ \neg\text{Def}(\text{prog3},t) & \rightarrow & \neg\text{Def}(\text{prog4},t) & \text{Rule NI2 (Figure 7)} \\ \text{Transp}(\text{prog3},e) & \Rightarrow & \text{Transp}(\text{prog4},e) & \text{Section 5.4} \\ \text{Transp}(\text{prog3},e) * \neg\text{Def}(\text{prog3},t) & \rightarrow & \text{Transp}(\text{prog4},e) * \neg\text{Def}(\text{prog4},t) & \text{Rules CI1, CI2 (Figure 7)} \end{array}$$

Recall that  $t$  is a new variable with respect to  $\text{prog2}$ . Hence, there cannot be any assignment to  $t$  in  $\text{prog2}$ . The transformation of  $\text{prog2}$  to  $\text{prog3}$  involves insertion of  $\text{ASSIGN}(t,e)$  at  $\text{newpoints}$  which are distinct from  $\text{orgavails3}$ . Hence,  $\text{AssignStmt}(\text{prog3},t,e) \rightarrow \text{AssignStmt}(\text{prog4},t,e)$ . Since RE is an isomorphic transformation, using  $\text{TTL}_{im}$  for the transformation of  $\text{prog3}$  to  $\text{prog4}$ , we have  $\varphi_3 \rightarrow \varphi_4$ .

Also  $\text{redund2} \Rightarrow \text{redund4}$ . We thus have the following simple derivation: (1)  $\text{redund2} \leftarrow \text{redund4}$ : by IR2, (2)  $\text{redund2} \supset \varphi_3$ : from Equation 5, (3)  $\varphi_3 \rightarrow \varphi_4$ : proved above, (4)  $\text{redund2} \rightarrow \varphi_4$ : by FC1, and (5)  $\text{redund4} \supset \varphi_4$ : from steps (1) and (4), MC. This establishes the verification condition (Equation 4) of the last transformation in the CSE specification.

## 7. Soundness of the logic

In Section 7.1, we define simulation, bisimulation, and a special case of weak bisimulation between Kripke structures. For each transformation primitive, we show in Section 7.2 what kind of simulation relation the transformation constructs between the input and the transformed Kripke structures. In Section 7.3, we prove that certain correlations between  $\text{CTL}_{bp}$  formulae hold between a pair of Kripke structures if a particular kind of simulation relation exists between them. The soundness of the transformation specific inference rules follows immediately from the results in Section 7.2 and Section 7.3.

### 7.1. Algebraic formulations of simulation relations

#### 7.1.1. Binary relations

Consider sets  $P$  and  $Q$  containing  $n$  and  $m$  elements respectively. We represent an ordered relation  $R \subseteq Q \times P$  by an  $(m \times n)$  boolean matrix. A subset  $X \subseteq P$  (or  $X \subseteq Q$ ) is denoted by a boolean vector of size  $n$  (or size  $m$ ). Let  $\mathcal{B}_n$  and  $\mathcal{B}_m$  denote the sets of all boolean vectors of sizes  $n$  and  $m$  respectively.

A relation  $R \subseteq Q \times P$  can also be considered as a function  $R: \mathcal{B}_n \rightarrow \mathcal{B}_m$ . For simplicity of presentation, we consider boolean vectors and column matrices interchangeably and do not distinguish between  $\mathcal{B}_n$  and  $\mathcal{B}_{(n,1)}$ . With the interpretation of a boolean matrix as a function over boolean vectors, the matrix multiplication  $R \cdot \vec{z}$  where  $\vec{z} \in \mathcal{B}_n$  can be seen as a function application. A matrix multiplication  $R_1 \cdot R_2$  can be considered as function composition  $(R_1 \circ R_2)$ . The transpose  $\widehat{R}$  of matrix  $R$  represents the inverse of relation  $R$  and is the function  $\widehat{R}: \mathcal{B}_m \rightarrow \mathcal{B}_n$ .

We now formalize various properties of a relation  $R$  in boolean matrix algebraic setting. Let  $\vec{z} \in \mathcal{B}_n$  and  $\vec{z}' \in \mathcal{B}_m$ . We abbreviate the one-to-one, one-to-many, many-to-one, and many-to-many properties of relations by 1-1, 1-m, m-1, m-m respectively.

$$\begin{aligned}
R \cdot \vec{1} &= \vec{1} && \text{(total)} \\
R \cdot \vec{1} &\leq \vec{1} && \text{(may not be total)} \\
\vec{z}' &= R \cdot \widehat{R} \cdot \vec{z}' && \text{(total but neither m-1 nor m-m)} \\
\vec{z}' &\leq R \cdot \widehat{R} \cdot \vec{z}' && \text{(total and possibly m-1 or m-m)} \\
R \cdot \widehat{R} \cdot \vec{z}' &\leq \vec{z}' && \text{(possibly not total and neither m-1 nor m-m)} \\
\overline{R \cdot \vec{z}} &= \overline{R \cdot \vec{z}} && \text{(total but neither 1-m nor m-m)} \\
\overline{R \cdot \vec{z}} &\leq \overline{R \cdot \vec{z}} && \text{(total and possibly 1-m or m-m)} \\
\overline{R \cdot \vec{z}} &\leq \overline{\widehat{R} \cdot \vec{z}} && \text{(possibly not total and neither 1-m nor m-m)}
\end{aligned} \tag{6}$$

The above list gives properties of relation  $R$  in its first argument. Similarly, we define properties of relation  $R$  in its second argument as follows.

$$\begin{aligned}
\widehat{R} \cdot \vec{1} &= \vec{1} && \text{(onto)} \\
\widehat{R} \cdot \vec{1} &\leq \vec{1} && \text{(may not be onto)} \\
\vec{z} &= \widehat{R} \cdot R \cdot \vec{z} && \text{(onto but neither 1-m nor m-m)} \\
\vec{z} &\leq \widehat{R} \cdot R \cdot \vec{z} && \text{(onto and possibly 1-m or m-m)} \\
\widehat{R} \cdot R \cdot \vec{z} &\leq \vec{z} && \text{(possibly not onto and neither 1-m nor m-m)} \\
\overline{\widehat{R} \cdot \vec{z}'} &= \overline{\widehat{R} \cdot \vec{z}'} && \text{(onto but neither m-1 nor m-m)} \\
\overline{\widehat{R} \cdot \vec{z}'} &\leq \overline{\widehat{R} \cdot \vec{z}'} && \text{(onto and possibly m-1 or m-m)} \\
\overline{\widehat{R} \cdot \vec{z}'} &\leq \overline{\widehat{R} \cdot \vec{z}'} && \text{(possibly not onto and neither m-1 nor m-m)}
\end{aligned} \tag{7}$$

### 7.1.2. Simulation relations between Kripke structures

Consider Kripke structures  $M = (G, P, L)$  and  $M' = (G', P', L')$  with  $A$  and  $A'$  as respective adjacency matrices. Let there be  $n$  nodes in  $M$  and  $n'$  nodes in  $M'$ . Let an  $(n' \times n)$  boolean matrix  $R$  be a relation between the nodes of  $M'$  and  $M$ . We use unprimed symbols for  $M$  and primed symbols for  $M'$ .

**Definition 14.** A relation  $R$  is a *simulation relation* between Kripke structures  $M$  and  $M'$  (denoted as  $M \triangleright_R M'$ ) if (1)  $R \cdot A \cdot \widehat{R} \leq A'$  and (2) for all  $\vec{z} \in \mathcal{B}_n$ ,  $\vec{z} \leq \widehat{R} \cdot R \cdot \vec{z}$ .

Condition (1) states that if  $p'$  and  $p$  are related by  $R$  ( $[R]_{p'}^p = 1$ ) and there exists an edge  $\langle p, q \rangle$  in  $M$  ( $[A]_p^q = 1$ ), and some  $q'$  is related to  $q$  ( $[R]_{q'}^q = [\widehat{R}]_q^{q'} = 1$ ) then there is an edge  $\langle p', q' \rangle$  ( $[A']_{p'}^{q'} = 1$ ). In other words, if there is an edge  $\langle p, q \rangle$  in  $M$  and  $p'$  and  $q'$  are related to  $p$  and  $q$  through  $R$ , then there exists an edge  $\langle p', q' \rangle$  in  $M'$ .

As discussed in Section 5, we model programs as Kripke structures. We consider local data flow properties as atomic propositions. The labeling of nodes by atomic propositions is determined by valuations of the properties. The standard formulations of simulation relations [CGP00] require related nodes to be labeled with same atomic propositions. Our aim is to model program transformations as Kripke transformations. Since program transformations can change statements, we shall not insist on equality of atomic labels.

From Equation (7), we know that condition (2) states that  $R$  should be an *onto* relation and it can possibly be one-to-many or many-to-many. Thus,  $M \triangleright_R M'$  if for *every* edge in  $M$ , there is a corresponding edge in  $M'$ .

**Definition 15.** A relation  $R$  is a *bisimulation relation* between Kripke structures  $M$  and  $M'$  (denoted as  $M \bowtie_R M'$ ) if  $M \triangleright_R M'$  and  $M' \triangleright_{\widehat{R}} M$ .

A relation  $R$  is a bisimulation between Kripke structures  $M$  and  $M'$  if  $R$  is a simulation between  $M$  and  $M'$  and its inverse i.e.  $\widehat{R}$  is a simulation between  $M'$  and  $M$ . Thus,  $R$  should be a total and onto relation such that for every edge in  $M$  there is a corresponding edge in  $M'$  and vice versa.

We now consider a special case of weak bisimulation called *one-step weak bisimulation*. The formulation is *asymmetric* in a sense that only one Kripke structure has internal nodes. Let  $M$  and  $M'$  be two Kripke structures such that only  $M'$  has internal nodes which do not correspond to any nodes in  $M$ . Unlike the standard formulations of weak bisimulation [BCG88, NV95], we require that no two internal nodes can have an edge between them. In other words, internal and visible nodes can only alternate. Let us denote the set of nodes in  $M$  by  $N$  and the set of nodes in  $M'$  by  $N'$ . Let us denote the set of internal nodes in  $M'$  by  $\eta'$ . The set of visible nodes is  $N' - \eta'$ .

We characterize a one-step weak bisimulation by three relations between the nodes of  $M$  and  $M'$ . An  $(n' \times n)$  relation  $R \subseteq (N' - \eta') \times N$  which relates visible nodes of the two Kripke structures. An  $(n' \times n)$  relation  $S \subseteq \eta' \times N$  which relates internal nodes of  $M'$  to visible nodes of  $M$  and an  $(n \times n')$  relation  $P \subseteq N \times \eta'$  which relates visible nodes of  $M$  to internal nodes of  $M'$ . Let  $n$  be the number of nodes in  $M$  and  $n'$  be the number of nodes in  $M'$ .

**Definition 16.** A triple  $\langle R, P, S \rangle$  is a *one-step weak bisimulation* between  $M$  and  $M'$  (denoted as  $M \preceq_{\langle R, P, S \rangle} M'$ ) if (1)  $R \cdot A \cdot \widehat{R} = (A' - R \cdot P - S \cdot \widehat{R}) + R \cdot P \cdot S \cdot \widehat{R}$ , (2)  $R \cdot P \leq A'$ , (3)  $S \cdot \widehat{R} \leq A'$ , (4)  $S \cdot \vec{1} = \widehat{P} \cdot \vec{1} = \vec{\eta}' = \vec{1} - R \cdot \vec{1}$ , (5)  $P \cdot R = 0$ , (6)  $\widehat{R} \cdot S = 0$ , (7) for all  $\vec{z} \in \mathcal{B}_n$ ,  $\vec{z} = \widehat{R} \cdot R \cdot \vec{z}$ , (8) for all  $\vec{z}' \in \mathcal{B}_{n'}$ ,  $\vec{z}' * \vec{\eta}' = R \cdot \widehat{R} \cdot (\vec{z}' * \vec{\eta}')$ , (9) for all  $\vec{z}' \in \mathcal{B}_{n'}$ ,  $\widehat{R} \cdot (\vec{z}' * \vec{\eta}') = \widehat{R} \cdot \vec{z}'$ , and (10)  $\vec{\eta}' * (A' \cdot \vec{\eta}' + \widehat{A}' \cdot \vec{\eta}') = \vec{0}$ .

Condition (1) states that if  $p'$  and  $p$  are related by  $R$  ( $[R]_{p'}^p = 1$ ) and there exists an edge  $\langle p, q \rangle$  in  $M$  ( $[A]_p^q = 1$ ), and some  $q'$  is related to  $q$  ( $[R]_q^{q'} = [\widehat{R}]_{q'}^q = 1$ ) then one of the following holds:

- (a) There is an edge  $\langle p', q' \rangle$  ( $[A']_{p'}^{q'} = 1$ ) and there does not exist any node  $r$  in  $M$  such that (i)  $p'$  is related to  $r$  by  $R$  ( $[R]_{p'}^r = 1$ ) and  $r$  is related to  $q'$  by  $P$  ( $[P]_r^{q'} = 1$ ) or (ii)  $p'$  is related to  $r$  by  $S$  ( $[S]_{p'}^r = 1$ ) and  $r$  is related to  $q'$  by  $\widehat{R}$  ( $[\widehat{R}]_r^{q'} = 1$ ).
- (b) There exists a node  $r'$  in  $M'$  such that  $p$  is related to  $r'$  through  $P$  ( $[P]_p^{r'} = 1$ ) and  $r'$  is related to  $q$  through  $S$  ( $[S]_{r'}^q = 1$ ). In other words, the condition states that the edge  $\langle p, q \rangle$  is split into two edges  $\langle p', r' \rangle$  and  $\langle r', q' \rangle$  (where  $r'$  is an internal node).

Condition (2) states that if  $p'$  and  $p$  are related by  $R$  ( $[R]_{p'}^p = 1$ ) and  $p$  is related to  $r'$  through  $P$  ( $[P]_p^{r'} = 1$ ) then there is an edge  $\langle p', r' \rangle$  ( $[A']_{p'}^{r'} = 1$ ). Similarly, condition (3) states that if  $r'$  and  $q$  are related by  $S$  and  $q'$  and  $q$  are related by  $R$  then there is an edge  $\langle r', q' \rangle$ .

Condition (4) states that the domain of  $S$  and the range of  $P$  is the set of internal nodes  $\eta'$  in  $M'$ . Further, the set of internal nodes is disjoint with the domain of  $R$ .

Condition (5) states that if  $p$  is related to  $p'$  by  $P$  ( $[P]_p^{p'} = 1$ ) then there does not exist any  $q$  related to  $p'$  by  $R$  ( $[R]_{p'}^q = 0$ , for all  $q \in N$ ). Similarly, condition (6) states that if  $p'$  is related to  $q$  by  $S$  ( $[S]_{p'}^q = 1$ ) then there does not exist any  $p$  such that  $p$  is related to  $p'$  by  $R$  ( $[R]_p^{p'} = [\widehat{R}]_p^{p'} = 0$ , for all  $p \in N$ ).

From Equations (7), we know that condition (7) states that  $R$  is an onto relation but neither one-to-many nor many-to-many. From Equation (6), we know that condition (8) states that  $R$  is a total relation on  $N' - \eta'$  but is neither many-to-one nor many-to-many. Further, condition (9) states that  $R$  does not relate any node in  $\eta'$  to a node node in  $N$ . Together, conditions (7), (8), and (9) state that  $R \subseteq (N' - \eta') \times N$  is a bijection.

Condition (10) states that no two internal nodes can be adjacent.

Thus,  $M \preceq_{\langle R, P, S \rangle} M'$  if for every edge  $\langle p, q \rangle$  in  $M$  there is a corresponding edge  $\langle p', q' \rangle$  in  $M'$  or there are two adjacent edges  $\langle p', r' \rangle$  and  $\langle r', q' \rangle$  such that  $p'$  and  $p$  as well as  $q'$  and  $q$  are related by  $R$ . Further,  $p$  and  $r'$  are related by  $P$  and  $r'$  and  $q$  are related by  $S$  where  $r'$  is an internal node in  $M'$ .

## 7.2. Primitive Kripke transformations and simulation relations

A primitive Kripke transformation constructs some kind of simulation relation between a transformed Kripke structure  $M'$  and the corresponding input Kripke structure  $M$ . We summarize these relations in Figure 9 and use them to correlate temporal properties across transformations in Section 7.3.

The relation  $C$  denotes the correspondence matrix of the primitive transformation (ref. Section 3). The relations  $N_P$  and  $N_S$  are relevant for node addition and node deletion transformations and are defined in



Relations \ Trans.	NS	NM	EA	ED	NA	ND	IM
$M \triangleright_C M'$	Yes	Yes	Yes	No	—	—	Yes
$M' \triangleright_{\widehat{C}} M$	Yes	No	No	Yes	—	—	Yes
$M \bowtie_C M'$	Yes	No	No	No	—	—	Yes
$M \preceq_{\langle C, N_P, N_S \rangle} M'$	—	—	—	—	Yes	No	—
$M' \preceq_{\langle \widehat{C}, N_P, N_S \rangle} M$	—	—	—	—	No	Yes	—

**Fig. 9.** Simulation relations between Kripke structures under primitive transformations

Sections 3.6 and 3.7 respectively. The temporal logic  $\text{CTL}_{\text{bp}}$  also consists of backward temporal operators which can be interpreted by inverting the edges of a Kripke structure. The simulation relations for inverted Kripke structures can be identified similarly.

We now prove the construction of simulation relations noted in Figure 9 for some representative transformations. Let the adjacency matrices of Kripke structures  $M$  and  $M'$  be  $A$  and  $A'$  respectively.

**Theorem 17.** If  $C$  is the correspondence matrix of a node splitting transformation from  $M$  to  $M'$  then  $C$  is a bisimulation relation between  $M'$  and  $M$ , that is,  $M \bowtie_C M'$ .

*Proof.* The proof is derived in two steps:

1. From Definition 1,  $C \cdot A \cdot \widehat{C} = A'$ . From Lemma 13, for all  $\vec{z} \in \mathcal{B}_n$ ,  $\vec{z} = \widehat{C} \cdot C \cdot \vec{z}$ . Hence,  $C$  is a simulation relation between  $M'$  and  $M$ , that is,  $M \triangleright_C M'$ .
2. We now show that  $\widehat{C}$  is a simulation relation between  $M$  and  $M'$ , that is,  $M' \triangleright_{\widehat{C}} M$ .
  - (1) We prove that  $\widehat{C} \cdot A' \cdot C = A$ .

$$\begin{array}{lll}
 C \cdot A \cdot \widehat{C} & = & A' \quad \text{Definition 1} \\
 \widehat{C} \cdot (C \cdot A \cdot \widehat{C}) & = & \widehat{C} \cdot A' \quad \text{monotonic} \\
 \widehat{C} \cdot C \cdot (A \cdot \widehat{C}) & = & \widehat{C} \cdot A' \quad \text{associative} \\
 A \cdot \widehat{C} & = & \widehat{C} \cdot A' \quad \text{Lemma 13} \\
 (A \cdot \widehat{C}) \cdot C & = & \widehat{C} \cdot A' \cdot C \quad \text{monotonic} \\
 A \cdot (\widehat{C} \cdot C) & = & \widehat{C} \cdot A' \cdot C \quad \text{associative} \\
 A & = & \widehat{C} \cdot A' \cdot C \quad \text{Lemma 13}
 \end{array}$$

- (2) As a special case of Lemma 13,  $\vec{z}' \leq C \cdot \widehat{C} \cdot \vec{z}'$ .

□

Similarly, we can prove that a node merging transformation constructs a simulation relation between  $M'$  and  $M$ . However, it does not create a bisimulation relation between  $M$  and  $M'$ . Suppose nodes  $p$  and  $q$  are merged into a node  $p'$  and there are edges  $\langle p, r \rangle$  and  $\langle q, s \rangle$ . If  $r'$  and  $s'$  are the nodes corresponding to  $r$  and  $s$  then in the transformed Kripke structure, we have edges  $\langle p', r' \rangle$  and  $\langle p', s' \rangle$  which correspond to edges  $\langle p, r \rangle$ ,  $\langle q, r \rangle$ ,  $\langle p, s \rangle$ , and  $\langle q, s \rangle$  of the input Kripke structure. Thus, the transformed Kripke structure has edges which do not correspond to any edges in the input graph.

**Theorem 18.** If  $C$  is the correspondence matrix of an edge addition transformation from  $M$  to  $M'$  then  $C$  is a simulation relation between  $M'$  and  $M$  i.e.  $M \triangleright_C M'$ .

*Proof.* (1) We prove that  $C \cdot A \cdot \widehat{C} < A'$ .

$$\begin{array}{lll}
A & < & A+E & \text{Definition 3} \\
C \cdot A & < & C \cdot (A+E) & \text{monotonic, } C \text{ is a bijection} \\
C \cdot A \cdot \widehat{C} & < & C \cdot (A+E) \cdot \widehat{C} & \text{monotonic} \\
C \cdot A \cdot \widehat{C} & < & C \cdot A \cdot \widehat{C} + C \cdot E \cdot \widehat{C} = A' & \text{distributive, Definition 3}
\end{array}$$

(2) As a special case of Lemma 13,  $\vec{z} = \widehat{C} \cdot C \cdot \vec{z}$ . □

An edge addition transformation adds at least one new edge to the transformed Kripke structure. Therefore,  $C$  cannot be a bisimulation relation between  $M'$  and  $M$ .

An edge deletion transformation deletes at least one edge from the input Kripke structure. Therefore  $\widehat{C}$  is a simulation relation between  $M$  and  $M'$ . Clearly,  $C$  is not a bisimulation between  $M$  and  $M'$ .

**Theorem 19.** If  $C$  is the correspondence matrix of a node addition transformation from  $M$  to  $M'$  and the relations  $N_P$  and  $N_S$  are as defined in Definition 5 then the triple  $\langle C, N_P, N_S \rangle$  is a one-step weak bisimulation relation between  $M'$  and  $M$  i.e.  $M \preceq_{\langle C, N_P, N_S \rangle} M'$ .

*Proof.* Let  $E$  be the edges to be split and  $\eta'$  be the set of new nodes in  $M'$ . The proof is derived as follows:

(1) We prove that  $C \cdot A \cdot \widehat{C} = (A' - C \cdot N_P - N_S \cdot \widehat{C}) + C \cdot N_P \cdot N_S \cdot \widehat{C}$ .

$$\begin{array}{ll}
(C \cdot A \cdot \widehat{C} - C \cdot E \cdot \widehat{C}) + C \cdot N_P + N_S \cdot \widehat{C} & = A' & \text{Definition 5} \\
C \cdot A \cdot \widehat{C} - C \cdot E \cdot \widehat{C} & = A' - C \cdot N_P - N_S \cdot \widehat{C} \\
(C \cdot A \cdot \widehat{C} - C \cdot E \cdot \widehat{C}) + C \cdot E \cdot \widehat{C} & = (A' - C \cdot N_P - N_S \cdot \widehat{C}) + C \cdot E \cdot \widehat{C} \\
C \cdot A \cdot \widehat{C} & = (A' - C \cdot N_P - N_S \cdot \widehat{C}) + C \cdot E \cdot \widehat{C} & E \leq A \\
C \cdot A \cdot \widehat{C} & = (A' - C \cdot N_P - N_S \cdot \widehat{C}) + C \cdot N_P \cdot N_S \cdot \widehat{C} & \text{Definition 5}
\end{array}$$

(2)  $C \cdot N_P \leq A'$  and (3)  $N_S \cdot \widehat{C} \leq A'$ .

(4)  $N_S \cdot \vec{1} = \widehat{N}_P \cdot \vec{1} = \vec{\eta}' = \vec{1} - C \cdot \vec{1}$ , (5)  $N_P \cdot C = 0$ , and (6)  $\widehat{C} \cdot N_S = 0$ .

(7) As a special case of Lemma 13,  $\vec{z} = \widehat{C} \cdot C \cdot \vec{z}$ .

(8) From the nature of  $C$ ,  $\vec{z}' * \vec{\eta}' = C \cdot \widehat{C} \cdot (\vec{z}' * \vec{\eta}')$ .

(9) From the nature of  $C$ ,  $\widehat{C} \cdot (\vec{z}' * \vec{\eta}') = \widehat{C} \cdot \vec{z}'$ .

(10) We prove that  $\vec{\eta}' * (A' \cdot \vec{\eta}' + \widehat{A}' \cdot \vec{\eta}') = \vec{0}$  as follows:

$$\begin{aligned}
\vec{\eta}' * A' \cdot \vec{\eta}' &= \vec{\eta}' * ((C \cdot A \cdot \widehat{C} - C \cdot E \cdot \widehat{C}) + C \cdot N_P + N_S \cdot \widehat{C}) \cdot \vec{\eta}' && \text{Definition 5} \\
&= \vec{\eta}' * C \cdot N_P \cdot \vec{\eta}' && \text{Definition 5} \\
&= (\vec{1} - C \cdot \vec{1}) * C \cdot N_P \cdot \vec{\eta}' && \text{Definition 5} \\
&\leq (\vec{1} * \vec{C} \cdot \vec{1}) * C \cdot \vec{1} = \vec{0}
\end{aligned}$$

$$\begin{aligned}
\vec{\eta}' * \widehat{A}' \cdot \vec{\eta}' &= \vec{\eta}' * ((C \cdot \widehat{A} \cdot \widehat{C} - C \cdot \widehat{E} \cdot \widehat{C}) + \widehat{N}_P \cdot \widehat{C} + C \cdot \widehat{N}_S) \cdot \vec{\eta}' && \text{Definition 5} \\
&= \vec{\eta}' * C \cdot \widehat{N}_S \cdot \vec{\eta}' && \text{Definition 5} \\
&= (\vec{1} - C \cdot \vec{1}) * C \cdot \widehat{N}_S \cdot \vec{\eta}' && \text{Definition 5} \\
&\leq (\vec{1} * \vec{C} \cdot \vec{1}) * C \cdot \vec{1} = \vec{0}
\end{aligned}$$

Hence,  $\vec{\eta}' * (A' \cdot \vec{\eta}' + \widehat{A}' \cdot \vec{\eta}') = \vec{0}$ . □

Similarly, we can show that if  $C$  is the correspondence matrix of a node deletion transformation from  $M$  to  $M'$  and the relations  $N_P$  and  $N_S$  are as defined in Definition 6 then the triple  $\langle \widehat{C}, N_P, N_S \rangle$  is a one-step weak bisimulation relation between  $M$  and  $M'$  i.e.  $M' \preceq_{\langle \widehat{C}, N_P, N_S \rangle} M$ .

The case of an isomorphic transformation is straightforward.

$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}'}{R \cdot A \cdot \vec{\varphi} \leq A' \cdot \vec{\varphi}'}$ <p>(SR:EX)</p>	$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')}$ <p>(SR:EU)</p>
$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\nu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \nu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')}$ <p>(SR:EW)</p>	$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}'}{R \cdot (\nu \vec{z}. \vec{\varphi} * A \cdot \vec{z}) \leq \nu \vec{z}'. \vec{\varphi}' * A' \cdot \vec{z}'}$ <p>(SR:EG)</p>

Fig. 10. Correlations of existential future properties under a simulation relation  $M \triangleright_R M'$

### 7.3. Correlating temporal properties

In this section, we prove correlations of temporal properties between Kripke structures that are related by various simulation relations. In Appendix A, we give boolean matrix algebraic semantics of  $\text{CTL}_{\text{bp}}$  formulae. Some temporal operators are given fixed point semantics. Since Kripke structures can have different number of nodes, in Appendix B, we define an ordering of functions defined over different vector spaces and prove relations between the least and greatest fixed points of such functions. For brevity, we consider correlations of future temporal properties under simulation and weak bisimulation relations. We refer the reader to [Kan07] for the complete set of correlations and their proofs.

#### 7.3.1. Existential future properties

The existential future  $\text{CTL}_{\text{bp}}$  operators are EX, EU, EW, EF, and EG where EF is a special case of EU.

Consider a simulation relation  $R$  between Kripke structures  $M$  and  $M'$ . For every edge in  $M$ , there is an edge in  $M'$ . Thus, we can correlate existential future properties between  $M$  and  $M'$ . The correlations are given in Figure 10 and their soundness is proved below.

**Theorem 20.** If  $R$  is a simulation relation between Kripke structures  $M$  and  $M'$  i.e.  $M \triangleright_R M'$  then the correlations between the future temporal properties given in Figure 10 are sound.

*Proof.* We first prove soundness of the rule SR:EX.

$R \cdot \vec{\varphi}$	$\leq$	$\vec{\varphi}'$		given
$R \cdot A \cdot \widehat{R} \cdot (R \cdot \vec{\varphi})$	$\leq$	$A' \cdot \vec{\varphi}'$		monotonic, Definition 14
$R \cdot A \cdot (\widehat{R} \cdot R \cdot \vec{\varphi})$	$\leq$	$A' \cdot \vec{\varphi}'$		associative
$R \cdot A \cdot \vec{\varphi}$	$\leq$	$A' \cdot \vec{\varphi}'$		Definition 14, monotonic

Note that the rules SR:EU, SR:EW, and SR:EG correlate fixed points of certain functions. To prove soundness of these rules, we use the following strategy. We first establish an ordering as per Definition 28 between the respective functions. We then use Lemmas 29 and 30 to correlate their fixed points. We demonstrate this strategy by proving soundness of the rule SR:EU. Let  $\vec{z}$  be a boolean vector defined on  $M$ .

$R \cdot \vec{z}$	$\leq$	$R \cdot \vec{z}$		identity
$R \cdot A \cdot \vec{z}$	$\leq$	$A' \cdot (R \cdot \vec{z})$		substitute $[\vec{z}/\vec{\varphi}, R \cdot \vec{z}/\vec{\varphi}']$ in SR:EX
$R \cdot \vec{\varphi}$	$\leq$	$\vec{\varphi}'$		given
$(R \cdot \vec{\varphi}) * (R \cdot A \cdot \vec{z})$	$\leq$	$\vec{\varphi}' * (A' \cdot (R \cdot \vec{z}))$		monotonic
$R \cdot (\vec{\varphi} * A \cdot \vec{z})$	$\leq$	$\vec{\varphi}' * (A' \cdot (R \cdot \vec{z}))$		distributive
$R \cdot \vec{\psi}$	$\leq$	$\vec{\psi}'$		given
$(R \cdot \vec{\psi}) + (R \cdot (\vec{\varphi} * A \cdot \vec{z}))$	$\leq$	$\vec{\psi}' + (\vec{\varphi}' * A' \cdot (R \cdot \vec{z}))$		monotonic
$R \cdot (\vec{\psi} + (\vec{\varphi} * A \cdot \vec{z}))$	$\leq$	$\vec{\psi}' + (\vec{\varphi}' * A' \cdot (R \cdot \vec{z}))$		distribute
$\lambda \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})$	$\sqsubseteq_R$	$\lambda \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')$		Definition 28
$R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z}))$	$\leq$	$\mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')$		Lemma 29

□

$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}'}{R \cdot A \cdot \vec{\varphi} \leq A' \cdot \vec{\varphi}'}$ <p>(WB:EX)</p>	$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')}$ <p>(WB:EU)</p>
$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\nu \vec{z}. \vec{\psi}' + (\vec{\varphi}' * A \cdot \vec{z})) \leq \nu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')}$ <p>(WB:EW)</p>	$\frac{R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}'}{R \cdot (\nu \vec{z}. \vec{\varphi}' * A \cdot \vec{z}) \leq \nu \vec{z}'. \vec{\varphi}' * A' \cdot \vec{z}'}$ <p>(WB:EG)</p>

Fig. 11. Correlations of existential future properties under a weak bisimulation relation  $M \preceq_{\langle R, P, S \rangle} M'$

**Theorem 21.** If  $\langle R, P, S \rangle$  is a weak bisimulation between Kripke structures  $M$  and  $M'$  i.e.  $M \preceq_{\langle R, P, S \rangle} M'$  and  $\vec{\eta}' = \vec{1} - R \cdot \vec{1}$  is the set of internal nodes of  $M'$  then the correlations between future temporal properties given in Figure 11 are sound.

*Proof.* We first prove soundness of the rule WB:EX.

$$(i) \quad \begin{array}{l} S \cdot (\widehat{R} \cdot \vec{\varphi}') \leq S \cdot \vec{1} = \vec{\eta}' \quad \text{by definition} \\ R \cdot P \cdot (S \cdot \widehat{R} \cdot \vec{\varphi}') \leq R \cdot P \cdot \vec{\eta}' \leq A' \cdot \vec{\eta}' \leq A' \cdot \vec{\varphi}' \quad \text{monotonic, Definition 16, given} \end{array}$$

The main derivation is as follows:

$$(ii) \quad \begin{array}{l} R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \text{given} \\ R \cdot A \cdot \widehat{R} \cdot (R \cdot \vec{\varphi}) \leq (A' + R \cdot P \cdot S \cdot \widehat{R}) \cdot \vec{\varphi}' \quad \text{Definition 16, monotonic} \\ R \cdot A \cdot (\widehat{R} \cdot R \cdot \vec{\varphi}) \leq A' \cdot \vec{\varphi}' + R \cdot P \cdot S \cdot \widehat{R} \cdot \vec{\varphi}' \quad \text{associative, distributive} \\ R \cdot A \cdot \vec{\varphi} \leq A' \cdot \vec{\varphi}' + R \cdot P \cdot (S \cdot \widehat{R} \cdot \vec{\varphi}') \quad \text{Definition 16, monotonic, associative} \\ R \cdot A \cdot \vec{\varphi} \leq A' \cdot \vec{\varphi}' \quad \text{from (i) above} \end{array}$$

Note that the rules WB:EU, WB:EW, and WB:EG correlate fixed points of certain functions. To prove soundness of these rules, we use the following strategy. Suppose the respective functions are  $f$  and  $f'$ . The functions are interpreted over different Kripke structures  $M$  and  $M'$  which are related by a one-step weak bisimulation. Hence,  $M'$  consists of internal nodes. Function  $f'$  does not distinguish between visible and internal nodes of  $M'$ . We therefore construct a function  $g$  which is similar to  $f'$  but *distinguishes between visible and internal nodes*. We show that  $f$  and  $g$  are related by an ordering as per Definition 28. We then correlate fixed points of  $f$  and  $g$  using Lemmas 29 and 30. Finally, we show that the required fixed point of  $g$  is smaller than the required fixed point of  $f'$ . We demonstrate this strategy by proving soundness of the rule WB:EU. For the rule WB:EU, let  $f = \lambda \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})$  and  $f' = \lambda \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')$ .

$$(i) \quad \begin{array}{l} R \cdot \vec{z} \leq R \cdot \vec{z} \quad \text{identity} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + R \cdot P \cdot S \cdot \widehat{R} \cdot (R \cdot \vec{z}) \quad \text{from (ii) of Thm. 21} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + (R \cdot P) \cdot S \cdot \widehat{R} \cdot (R \cdot \vec{z}) \quad \text{associative} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + A' \cdot S \cdot \widehat{R} \cdot (R \cdot \vec{z}) \quad \text{Def. 16, monotonic} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + A' \cdot S \cdot \widehat{R} \cdot (\vec{1} * R \cdot \vec{z}) \quad \text{identity} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + A' \cdot (S \cdot \widehat{R} \cdot \vec{1} * S \cdot \widehat{R} \cdot R \cdot \vec{z}) \quad \text{distributive} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + A' \cdot (\vec{\eta}' * (S \cdot \widehat{R}) \cdot R \cdot \vec{z}) \quad \text{Def. 16, associative} \\ R \cdot A \cdot \vec{z} \leq A' \cdot (R \cdot \vec{z}) + A' \cdot (\vec{\eta}' * A' \cdot R \cdot \vec{z}) \quad \text{Def. 16, monotonic} \end{array}$$

$$(ii) \quad \begin{array}{l} R \cdot (\vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \vec{\psi}' + (\vec{\varphi}' * (A' \cdot R \cdot \vec{z} + A' \cdot (\vec{\eta}' * A' \cdot R \cdot \vec{z}))) \quad \text{monotonic} \\ \lambda \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z}) \sqsubseteq_R \underbrace{\lambda \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * (A' \cdot \vec{z}' + A' \cdot (\vec{\eta}' * A' \cdot \vec{z}')))}_{g} \quad \text{Def. 28} \end{array}$$

$$(iii) \quad R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * (A' \cdot \vec{z}' + A' \cdot (\vec{\eta}' * A' \cdot \vec{z}')))) \quad \text{Lemma 29}$$

$\frac{\widehat{R} \cdot R \cdot \vec{z} \leq \vec{z} \quad \overline{R \cdot \vec{z}} = R \cdot \overline{\vec{z}} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}'}{R \cdot \overline{A \cdot \vec{\varphi}} \leq \overline{A' \cdot \vec{\varphi}'}} \quad \text{(SR:AX)}$	$\frac{\widehat{R} \cdot R \cdot \vec{z} \leq \vec{z} \quad \overline{R \cdot \vec{z}} = R \cdot \overline{\vec{z}} \quad R \cdot \vec{\varphi} \leq \vec{\varphi} \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \overline{\vec{z}})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \overline{\vec{z}'})} \quad \text{(SR:AU)}$
$\frac{\widehat{R} \cdot R \cdot \vec{z} \leq \vec{z} \quad \overline{R \cdot \vec{z}} = R \cdot \overline{\vec{z}} \quad R \cdot \vec{\varphi} \leq \vec{\varphi} \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\nu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \overline{\vec{z}})) \leq \nu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \overline{\vec{z}'})} \quad \text{(SR:AW)}$	$\frac{\widehat{R} \cdot R \cdot \vec{z} \leq \vec{z} \quad \overline{R \cdot \vec{z}} = R \cdot \overline{\vec{z}} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}'}{R \cdot (\nu \vec{z}. \vec{\varphi} * A \cdot \overline{\vec{z}}) \leq \nu \vec{z}'. \vec{\varphi}' * A' \cdot \overline{\vec{z}'}} \quad \text{(SR:AG)}$

**Fig. 12.** Correlations of universal future properties under a simulation relation  $M' \triangleright_{\widehat{R}} M$

We now show that  $\mu \vec{z}'. g(\vec{z}') \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \overline{\vec{z}'})$ . Let  $\vec{q}' = \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \overline{\vec{z}'})$ .

$$\begin{array}{llll}
\vec{\eta}' & \leq & \vec{\varphi}' & \text{given} \\
\vec{\eta}' * A' \cdot \vec{q}' & \leq & \vec{\varphi}' * A' \cdot \vec{q}' \leq \vec{q}' & \text{monotonic; Def. of } \vec{q}' \\
A' \cdot (\vec{\eta}' * A' \cdot \vec{q}') & \leq & A' \cdot \vec{q}' & \text{monotonic} \\
A' \cdot \vec{q}' + A' \cdot (\vec{\eta}' * A' \cdot \vec{q}') & = & A' \cdot \vec{q}' & \\
(iv) \quad \vec{\psi}' + (\vec{\varphi}' * (A' \cdot \vec{q}' + A' \cdot (\vec{\eta}' * A' \cdot \vec{q}'))) & = & \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{q}') = \vec{q}' & \text{monotonic; Def. of } \vec{q}'
\end{array}$$

Thus,  $\vec{q}'$  is a fixed point of  $g$ . Hence,  $\mu \vec{z}'. g(\vec{z}') \leq \vec{q}'$ . From (iii) and (iv), by transitivity, we get the following result:  $R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \overline{\vec{z}})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \overline{\vec{z}'})$ .  $\square$

### 7.3.2. Universal future properties

The universal future CTL<sub>bp</sub> operators are AX, AU, AW, AF, and AG where AF is a special case of AU.

Consider a simulation relation  $\widehat{R}$  between Kripke structures  $M'$  and  $M$  where  $M' \triangleright_{\widehat{R}} M$ . The set of outgoing edges of a node in  $M$  is a superset of the set of outgoing edges of a corresponding node in  $M'$ . Since  $M'$  is a Kripke structure, the set of outgoing edges for any node is non-empty. We can correlate universal future properties between  $M$  and  $M'$ . The correlations are given in Figure 12 and their soundness is proved below.

**Theorem 22.** If  $\widehat{R}$  is a simulation relation between Kripke structures  $M'$  and  $M$  i.e.  $M' \triangleright_{\widehat{R}} M$  then the correlations between future temporal properties given in Figure 12 are sound.

*Proof.* We first determine the nature of the simulation relation from its definition and the premises common in the inference rules in Figure 12.

From Definition 14, we know that  $\vec{z}' \leq R \cdot \widehat{R} \cdot \vec{z}'$ . Thus,  $R$  is total and possibly many-to-one or many-to-many. From (6), we know that  $\overline{R \cdot \vec{z}} = R \cdot \overline{\vec{z}}$  states that  $R$  is total but neither one-to-many nor many-to-many. From (7), we know that  $\widehat{R} \cdot R \cdot \vec{z} \leq \vec{z}$  states that  $R$  is possibly not onto and neither one-to-many nor many-to-many. Thus,  $R$  is a total and either a one-to-one or a many-to-one relation.

Further, from Definition 14,  $\widehat{R} \cdot A' \cdot R \leq A$ . Hence, the set of outgoing edges of a node in  $M$  is a superset of the set of outgoing edges of a corresponding node in  $M'$ .

The proof of the rule SR:AX is as follows:

$$\begin{array}{lll}
R \cdot \vec{\varphi} & \leq & \vec{\varphi}' & \text{given} \\
\overline{\vec{\varphi}'} & \leq & \overline{R \cdot \vec{\varphi}} = R \cdot \overline{\vec{\varphi}} & \text{negation, given} \\
\widehat{R} \cdot \vec{\varphi}' & \leq & \widehat{R} \cdot R \cdot \overline{\vec{\varphi}} \leq \overline{\vec{\varphi}} & \text{monotonicity, given} \\
\widehat{R} \cdot A' \cdot R \cdot (\widehat{R} \cdot \vec{\varphi}') & \leq & A \cdot \overline{\vec{\varphi}} & \text{Definition 14, monotonic} \\
\widehat{R} \cdot A' \cdot (R \cdot \widehat{R} \cdot \vec{\varphi}') & \leq & A \cdot \overline{\vec{\varphi}} & \text{associative} \\
R \cdot (\widehat{R} \cdot A' \cdot \vec{\varphi}') & \leq & R \cdot (A \cdot \overline{\vec{\varphi}}) & \text{Definition 14, monotonic} \\
R \cdot \widehat{R} \cdot (A' \cdot \vec{\varphi}') & \leq & R \cdot A \cdot \overline{\vec{\varphi}} & \text{associative} \\
A' \cdot \vec{\varphi}' & \leq & \overline{R \cdot A \cdot \overline{\vec{\varphi}}} & \text{Definition 14} \\
\overline{R \cdot A \cdot \overline{\vec{\varphi}}} & \leq & \overline{A' \cdot \vec{\varphi}'} & \text{negation} \\
R \cdot \overline{A \cdot \overline{\vec{\varphi}}} & \leq & \overline{A' \cdot \vec{\varphi}'} & \text{given}
\end{array}$$

The proof of the rule SR:AU is as follows:

Let  $\vec{z}$  be a boolean vector defined on  $M$ .

$$\begin{array}{lll}
R \cdot \vec{z} & \leq & R \cdot \vec{z} & \text{identity} \\
R \cdot \overline{A \cdot \vec{z}} & \leq & \overline{A' \cdot (R \cdot \vec{z})} & [\vec{z}/\vec{\varphi}, R \cdot \vec{z}/\vec{\varphi}'] \text{ in Theorem 22} \\
R \cdot \vec{\varphi} & \leq & \vec{\varphi}' & \text{given} \\
(R \cdot \vec{\varphi}) * \overline{(R \cdot A \cdot \vec{z})} & \leq & \overline{\vec{\varphi}' * (A' \cdot (R \cdot \vec{z}))} & \text{monotonic} \\
R \cdot (\vec{\varphi} * A \cdot \vec{z}) & \leq & \overline{\vec{\varphi}' * (A' \cdot (R \cdot \vec{z}))} & \text{distributive} \\
R \cdot \vec{\psi} & \leq & \vec{\psi}' & \text{assumption}
\end{array}$$

$$\begin{array}{lll}
(R \cdot \vec{\psi}) + \left( \overline{R \cdot (\vec{\varphi} * A \cdot \vec{z})} \right) & \leq & \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot (R \cdot \vec{z}))} & \text{monotonic} \\
R \cdot (\vec{\psi} + \overline{(\vec{\varphi} * A \cdot \vec{z})}) & \leq & \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot (R \cdot \vec{z}))} & \text{distributive} \\
(i) \quad \lambda \vec{z}. \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot \vec{z})} & \sqsubseteq_R & \lambda \vec{z}'. \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot \vec{z}')} & \text{Definition 28} \\
R \cdot (\mu \vec{z}. \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot \vec{z})}) & \leq & \mu \vec{z}'. \vec{\psi}' + \overline{(\vec{\varphi}' * A' \cdot \vec{z}')} & \text{Lemma 29}
\end{array}$$

The proofs of SR:AW and SR:AG can be derived similarly.  $\square$

Consider a one-step weak bisimulation  $M \preceq_{\langle R, P, S \rangle} M'$  where  $\eta'$  denotes the set of internal nodes in  $M'$ . The correlations between universal future properties of  $M$  and  $M'$  are given in Figure 13.

We prove two supporting lemmas before giving the proof of soundness of the inference rules.

**Lemma 23.** If  $M \preceq_{\langle R, P, S \rangle} M'$  then  $R \cdot \widehat{R} \cdot \vec{z}' \leq \vec{z}'$ .

*Proof.* From Definition 16,  $\widehat{R} \cdot \vec{z}' = \widehat{R} \cdot (\vec{z}' * \overline{\eta'})$ . Multiply both sides by  $R$ . Again, from Definition 16,  $R \cdot \widehat{R} \cdot \vec{z}' = R \cdot \widehat{R} \cdot (\vec{z}' * \overline{\eta'}) = \vec{z}' * \overline{\eta'}$ . Further,  $\vec{z}' * \overline{\eta'} \leq \vec{z}'$ . Hence,  $R \cdot \widehat{R} \cdot \vec{z}' \leq \vec{z}'$ .  $\square$

**Lemma 24.** If  $M \preceq_{\langle R, P, S \rangle} M'$  then  $\widehat{R} \cdot A' \cdot R + \widehat{R} \cdot R \cdot P \cdot S \cdot \widehat{R} \cdot R = \widehat{R} \cdot (R \cdot A \cdot \widehat{R}) \cdot R$ .

*Proof.*

$$\begin{array}{lll}
(A' - R \cdot P - S \cdot \widehat{R}) + R \cdot P \cdot S \cdot \widehat{R} & = & R \cdot A \cdot \widehat{R} & \text{Definition 16} \\
\widehat{R} \cdot ((A' - R \cdot P - S \cdot \widehat{R}) + R \cdot P \cdot S \cdot \widehat{R}) \cdot R & = & \widehat{R} \cdot (R \cdot A \cdot \widehat{R}) \cdot R & \text{monotonic} \\
\widehat{R} \cdot A' \cdot R + \widehat{R} \cdot R \cdot P \cdot S \cdot \widehat{R} \cdot R & = & \widehat{R} \cdot (R \cdot A \cdot \widehat{R}) \cdot R & \text{distributive, Definition 16}
\end{array}$$

$\square$

**Theorem 25.** If  $\langle R, P, S \rangle$  is a weak bisimulation between Kripke structures  $M$  and  $M'$  i.e.  $M \preceq_{\langle R, P, S \rangle} M'$  and  $\vec{\eta}' = \vec{1} - R \cdot \vec{1}$  is the set of internal nodes of  $M'$  then the correlations between future temporal properties given in Figure 13 are sound.

$$\begin{array}{c}
\frac{\widehat{R} \cdot \vec{z}' \leq \widehat{R} \cdot \vec{z}' \quad \overline{R \cdot \vec{z} * \vec{\eta}'} = R \cdot \vec{z} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}'}{R \cdot A \cdot \vec{\varphi} \leq A \cdot \vec{\varphi}'} \\
\text{(WB:AX)} \\
\frac{\widehat{R} \cdot \vec{z}' \leq \widehat{R} \cdot \vec{z}' \quad \overline{R \cdot \vec{z} * \vec{\eta}'} = R \cdot \vec{z} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \mu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')} \\
\text{(WB:AU)} \\
\frac{\widehat{R} \cdot \vec{z}' \leq \widehat{R} \cdot \vec{z}' \quad \overline{R \cdot \vec{z} * \vec{\eta}'} = R \cdot \vec{z} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}' \quad R \cdot \vec{\psi} \leq \vec{\psi}'}{R \cdot (\nu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})) \leq \nu \vec{z}'. \vec{\psi}' + (\vec{\varphi}' * A' \cdot \vec{z}')} \\
\text{(WB:AW)} \\
\frac{\widehat{R} \cdot \vec{z}' \leq \widehat{R} \cdot \vec{z}' \quad \overline{R \cdot \vec{z} * \vec{\eta}'} = R \cdot \vec{z} \quad R \cdot \vec{\varphi} \leq \vec{\varphi}' \quad \vec{\eta}' \leq \vec{\varphi}'}{R \cdot (\nu \vec{z}. \vec{\varphi} * A \cdot \vec{z}) \leq \nu \vec{z}'. \vec{\varphi}' * A' \cdot \vec{z}'} \\
\text{(WB:AG)}
\end{array}$$

**Fig. 13.** Correlations of universal future properties under a weak bisimulation relation  $M \leq_{(R,P,S)} M'$

*Proof.* We first determine the nature of the relation  $R$  from the definition of weak bisimulation relations and the premises common in the rules of Figure 13.

From Definition 16, we know that  $\vec{z} = \widehat{R} \cdot R \cdot \vec{z}$ . Thus,  $R$  is onto but neither one-to-many nor many-to-many ref. (7). From (7), we know that  $\widehat{R} \cdot \vec{z}' \leq \widehat{R} \cdot \vec{z}'$  states that  $R$  is onto and possibly many-to-one or many-to-many. Further,  $\overline{R \cdot \vec{z} * \vec{\eta}'} = R \cdot \vec{z}$  implies that  $\overline{R \cdot \vec{z}} \geq R \cdot \vec{z}$ . From (6), we know that  $\overline{R \cdot \vec{z}} \geq R \cdot \vec{z}$  states that  $R$  is not total and neither one-to-many nor many-to-many. Thus,  $R$  is partial, onto, and either one-to-one or many-to-one.

Further, from Definition 16,  $R \cdot A \cdot \widehat{R} = (A' - R \cdot P - S \cdot \widehat{R}) + R \cdot P \cdot S \cdot \widehat{R}$ . Hence, for every edge  $\langle p, q \rangle$  in  $M$ , in  $M'$  either (a) there is an edge  $\langle p', q' \rangle$  where  $p'$  and  $p$  as  $q'$  and  $q$  are related by  $R$  or (b) a pair of adjacent edges  $\langle p', r' \rangle$  and  $\langle r', q' \rangle$  s.t.  $p'$  and  $p$  as well as  $q'$  and  $q$  are related by  $R$  and  $r'$  is an internal node. Similarly, for every edge or a pair of adjacent edges in  $M'$  there is an edge in  $M$ .

The proof of the rule WB:AX is as follows:

$$\begin{array}{lll}
R \cdot \vec{\varphi} & \leq & \vec{\varphi}' & \text{given} \\
\vec{\eta}' & \leq & \vec{\varphi}' & \text{given} \\
R \cdot \vec{\varphi} + \vec{\eta}' & \leq & \vec{\varphi}' & \text{monotonic, identity} \\
\vec{\varphi}' & \leq & \overline{R \cdot \vec{\varphi} * \vec{\eta}'} = R \cdot \vec{\varphi} & \text{monotonic, given} \\
\vec{\varphi}' * \vec{\eta}' & \leq & R \cdot \vec{\varphi} & \vec{\varphi}' \leq \vec{\eta}' \text{ hence } \vec{\varphi}' * \vec{\eta}' = \vec{\varphi}' \\
\widehat{R} \cdot (\vec{\varphi}' * \vec{\eta}') & \leq & \widehat{R} \cdot R \cdot \vec{\varphi} = \vec{\varphi} & \text{monotonic, Definition 16} \\
\widehat{R} \cdot A' \cdot R \cdot (\widehat{R} \cdot (\vec{\varphi}' * \vec{\eta}')) & \leq & \widehat{R} \cdot (R \cdot A \cdot \widehat{R}) \cdot R \cdot \vec{\varphi} & \text{Lemma 24, monotonic} \\
\widehat{R} \cdot A' \cdot (R \cdot \widehat{R} \cdot (\vec{\varphi}' * \vec{\eta}')) & \leq & \widehat{R} \cdot R \cdot A \cdot (\widehat{R} \cdot R \cdot \vec{\varphi}) & \text{associative} \\
\widehat{R} \cdot A' \cdot (\vec{\varphi}' * \vec{\eta}') & \leq & \widehat{R} \cdot R \cdot (A \cdot \vec{\varphi}) & \text{Definition 16, associative} \\
\widehat{R} \cdot A' \cdot \vec{\varphi}' & \leq & A \cdot \vec{\varphi} & \text{Definition 16, } \vec{\varphi}' * \vec{\eta}' = \vec{\varphi}' \\
A \cdot \vec{\varphi} & \leq & \widehat{R} \cdot A' \cdot \vec{\varphi}' \leq \widehat{R} \cdot A' \cdot \vec{\varphi}' & \text{negation, given} \\
R \cdot A \cdot \vec{\varphi} & \leq & R \cdot \widehat{R} \cdot A' \cdot \vec{\varphi}' & \text{monotonic} \\
R \cdot A \cdot \vec{\varphi} & \leq & A' \cdot \vec{\varphi}' & \text{Lemma 23}
\end{array}$$

The proof of the rule WB:AU is as follows:

$$\begin{array}{ll}
\overline{R \cdot \bar{z} * \bar{\eta}'} & = R \cdot \bar{z} & \text{given} \\
\widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) & = \widehat{R} \cdot R \cdot \bar{z} = \bar{z} & \text{Def.16} \\
(\widehat{R} \cdot A' \cdot R + \widehat{R} \cdot R \cdot P \cdot S \cdot \widehat{R} \cdot R) \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) & = \widehat{R} \cdot R \cdot A \cdot \widehat{R} \cdot R \cdot \bar{z} & \text{Lemma 24} \\
\widehat{R} \cdot A' \cdot R \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot R \cdot P \cdot S \cdot \widehat{R} \cdot R \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) & = A \cdot \bar{z} & \text{distributive, Def. 16} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot (R \cdot P) \cdot S \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) & \leq A \cdot \bar{z} & \text{assoc., Definition 16} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot (R \cdot P) \cdot S \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) & \leq A \cdot \bar{z} & \text{Definition 16} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot (R \cdot P) \cdot S \cdot \widehat{R} \cdot (\overline{1 * (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \leq A \cdot \bar{z} & \text{identity} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot (R \cdot P) \cdot (S \cdot \widehat{R} \cdot \overline{1 * S \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \leq A \cdot \bar{z} & \text{distributive} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) + \widehat{R} \cdot (R \cdot P) \cdot (\overline{\eta' * (S \cdot \widehat{R} \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})})}) & \leq A \cdot \bar{z} & \text{Definition 16} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * \widehat{R} \cdot A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \geq A \cdot \bar{z} & \text{negation} \\
\widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * \widehat{R} \cdot A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \geq A \cdot \bar{z} & \text{given} \\
R \cdot \widehat{R} \cdot A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * R \cdot \widehat{R} \cdot A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \geq R \cdot A \cdot \bar{z} & \text{mono., distributive} \\
A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \geq R \cdot A \cdot \bar{z} & \text{Lemma 23} \\
(i) \ A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})}) & \geq R \cdot A \cdot \bar{z} & \overline{R \cdot \bar{z}} \geq R \cdot \bar{z} \\
\overline{\psi' + (\overline{\varphi' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\overline{R \cdot \bar{z} * \bar{\eta}'})})})} & \geq R \cdot (\overline{\psi' + (\overline{\varphi' * A \cdot \bar{z}})}) & \text{monotonic} \\
(ii) \ \underbrace{\lambda z'. \overline{\psi' + (\overline{\varphi' * A' \cdot (\overline{z' * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\overline{z' * \bar{\eta}'})})})}}_{g} & \sqsupseteq_R \lambda z'. \overline{\psi' + (\overline{\varphi' * A \cdot \bar{z}})} & \text{Definition 28} \\
(iii) \ \mu z'. \overline{\psi' + (\overline{\varphi' * A' \cdot (\overline{z' * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\overline{z' * \bar{\eta}'})})})} & \geq R \cdot (\mu z'. \overline{\psi' + (\overline{\varphi' * A \cdot \bar{z}})}) & \text{Lemma 29}
\end{array}$$

We now show that  $\mu z'. g(z') \leq \mu z'. \overline{\psi' + \overline{\varphi' * A' \cdot \bar{z}'}}$ . Let  $\bar{q}' = \mu z'. \overline{\psi' + \overline{\varphi' * A' \cdot \bar{z}'}}$ .

$$\begin{array}{ll}
\bar{q}' & \geq \overline{\varphi' * A' \cdot \bar{q}'} & \text{Definition of } \bar{q}' \\
\bar{q}' & \leq \overline{\varphi' + A' \cdot \bar{q}'} & \text{negation} \\
\bar{q}' * \bar{\eta}' & \leq (\overline{\eta' * \varphi'}) + (\overline{\eta' * A' \cdot \bar{q}'}) & \text{monotonic} \\
\bar{q}' * \bar{\eta}' & \leq \overline{0 + (\eta' * A' \cdot \bar{q}')} & \bar{\eta}' \leq \overline{\varphi'} \\
(iv) \ \bar{q}' * \bar{\eta}' & \leq \overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')} & \text{Definition 16}
\end{array}$$

Further,

$$\begin{array}{ll}
(\overline{\bar{q}' * \bar{\eta}'} + \overline{\bar{q}' * \bar{\eta}'}) & = \bar{q}' \\
(\overline{\bar{q}' * \bar{\eta}'} + \overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')} & \geq \bar{q}' & \text{from (iv)} \\
A' \cdot (\overline{\bar{q}' * \bar{\eta}'}) + A' \cdot (\overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')} & \geq A' \cdot \bar{q}' & \text{monotonic} \\
A' \cdot (\overline{\bar{q}' * \bar{\eta}'}) * A' \cdot (\overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')} & \leq A' \cdot \bar{q}' & \text{negate; distribute} \\
\overline{\psi' + (\overline{\varphi' * A' \cdot (\bar{q}' * \bar{\eta}')} * A' \cdot (\overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')}))} & \leq \overline{\psi' + (\overline{\varphi' * A' \cdot \bar{q}'})} & \text{monotonic} \\
\overline{\psi' + (\overline{\varphi' * A' \cdot (\bar{q}' * \bar{\eta}')} * A' \cdot (\overline{\eta' * A' \cdot (\bar{q}' * \bar{\eta}')}))} & \leq \bar{q}' & \text{Def. of } \bar{q}'
\end{array}$$

Thus,  $\bar{q}'$  is a post-fixed point of  $g$ . Hence,  $\mu z'. g(z') \leq \bar{q}'$ . From (iii) above, by transitivity,

$$R \cdot (\mu z'. \overline{\psi' + (\overline{\varphi' * A \cdot \bar{z}})}) \leq \mu z'. \overline{\psi' + (\overline{\varphi' * A' \cdot \bar{z}'})}$$

The proofs of WB:AW and WB:AG can be derived similarly.  $\square$



## 8. Conclusions

Program transformations are used routinely in many software development and maintenance activities. Temporal logics provide an expressive and powerful framework to specify program properties. In this paper, we have considered the problem of correlating temporal properties across program transformations. We have developed a logic called temporal transformation logic and presented inference rules for a comprehensive set of primitive program transformations. This formulation enables us to deductively verify soundness of program transformations when the soundness conditions of individual transformations can be captured as temporal logic formulae. In particular, we have considered its application to verification of compiler optimizations. We have made novel use of boolean matrix algebra in defining the transformation primitives and proving soundness of the inference rules of the logic.

Future work involves applying the logic to verification problems arising in software engineering due to the use of program transformations. On the theoretical front, we plan to study completeness of the logic and its extension to CTL\* and modal mu-calculus. The present formulations are applicable to intra-procedural temporal properties only. In the inter-procedural framework, we need to consider more expressive models of programs like recursive state machines [ABE<sup>+</sup>05] or pushdown systems [RHS95]. For inter-procedural analysis, temporal logics have been extended with matching calls and returns [AEM04]. Extending the logic to inter-procedural setting is a challenging research direction.

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## A. Computational tree logic with branching past

The syntax of  $\text{CTL}_{\text{bp}}$  formulae is as follows:

$$\begin{aligned} \varphi := & \top \mid \perp \mid p \mid \neg\varphi \mid \varphi \supset \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \\ & \text{EX}(\varphi) \mid \text{EU}(\varphi, \varphi) \mid \text{EW}(\varphi, \varphi) \mid \text{EF}(\varphi) \mid \text{EG}(\varphi) \mid \\ & \text{EY}(\varphi) \mid \text{ES}(\varphi, \varphi) \mid \text{EP}(\varphi) \mid \text{EH}(\varphi) \mid \\ & \text{AX}(\varphi) \mid \text{AU}(\varphi, \varphi) \mid \text{AW}(\varphi, \varphi) \mid \text{AF}(\varphi) \mid \text{AG}(\varphi) \mid \\ & \text{AY}(\varphi) \mid \text{AS}(\varphi, \varphi) \mid \text{AP}(\varphi) \mid \text{AH}(\varphi) \end{aligned}$$

where  $\top$  and  $\perp$  are special propositions denoting true and false values and  $p$  is an atomic proposition.

$\text{CTL}_{\text{bp}}$  is interpreted over models with branching past and branching future. Future is infinite whereas past is finite. Models of  $\text{CTL}_{\text{bp}}$  formulae are nodes of Kripke structures. For a formula  $\varphi$  and a node  $i$  of a Kripke structure  $M = (G, P, L)$ , we write  $M, i \models \varphi$  to denote that the formula  $\varphi$  holds at the node  $i$ . With the boolean matrix algebraic semantics,  $M, i \models \varphi$  iff  $[\vec{\varphi}]_i = 1$ .

Consider a valuation function  $V$  from  $\text{CTL}_{\text{bp}}$  formulae to boolean vectors and denote  $V(\varphi)$  by  $\vec{\varphi}$ . The valuation of an atomic proposition  $p$  is  $V(p) = L(p) = \vec{p}$ .

To define semantics of  $\text{CTL}_{\text{bp}}$  operators, we require the notion of fixed points over boolean vector domains. Consider a complete lattice  $D = \langle \mathcal{B}, \leq \rangle$  where  $\mathcal{B} = \{0, 1\}$  is a set of boolean values with a partial order  $0 \leq 1$ . A complete lattice  $D_n = \langle \mathcal{B}_n, \leq_n \rangle$  is a cartesian product of  $n$   $D$ -lattices. Consider a monotonic function  $f : D^n \rightarrow D^n$ . The least fixed point of  $f$  is denoted by  $\mu \vec{z}. f(\vec{z})$  and the greatest fixed point of  $f$  is denoted by  $\nu \vec{z}. f(\vec{z})$ . As a convention, we do not write the subscript of the partial order  $\leq$  when used for vectors.

**Definition 26.** Consider a Kripke structure  $M$ . Let  $V(\top) = \vec{1}$ ,  $V(\perp) = \vec{0}$ , and  $V(p) = \vec{p}$ .

$V(\neg\varphi)$	$\triangleq \overline{\vec{\varphi}}$	negation
$V(\varphi \supset \psi)$	$\triangleq \overline{\vec{\varphi}} + \vec{\psi}$	implication
$V(\varphi \vee \psi)$	$\triangleq \vec{\varphi} + \vec{\psi}$	disjunction
$V(\varphi \wedge \psi)$	$\triangleq \vec{\varphi} * \vec{\psi}$	conjunction
$V(\text{EX}(\varphi))$	$\triangleq A \cdot \vec{\varphi}$	Exists neXt
$V(\text{EU}(\varphi, \psi))$	$\triangleq \mu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})$	Exists Until
$V(\text{EW}(\varphi, \psi))$	$\triangleq \nu \vec{z}. \vec{\psi} + (\vec{\varphi} * A \cdot \vec{z})$	Exists Weak until
$V(\text{EF}(\varphi))$	$\triangleq \mu \vec{z}. \vec{\varphi} + (\vec{1} * A \cdot \vec{z})$	Exists in Future
$V(\text{EG}(\varphi))$	$\triangleq \nu \vec{z}. \vec{\varphi} * A \cdot \vec{z}$	Exists Globally
$V(\text{AX}(\varphi))$	$\triangleq \overline{A \cdot \vec{\varphi}}$	forAll neXt
$V(\text{AU}(\varphi, \psi))$	$\triangleq \mu \vec{z}. \vec{\psi} + (\vec{\varphi} * \overline{A \cdot \vec{z}})$	forAll Until
$V(\text{AW}(\varphi, \psi))$	$\triangleq \nu \vec{z}. \vec{\psi} + (\vec{\varphi} * \overline{A \cdot \vec{z}})$	forAll Weak until
$V(\text{AF}(\varphi))$	$\triangleq \mu \vec{z}. \vec{\varphi} + (\vec{1} * \overline{A \cdot \vec{z}})$	forAll in Future
$V(\text{AG}(\varphi))$	$\triangleq \nu \vec{z}. \vec{\varphi} * \overline{A \cdot \vec{z}}$	forAll Globally

To define semantics of past operators, we consider the transpose of the adjacency matrix  $A$  which corresponds to the inverted edges of the Kripke structure.

**Definition 27.** Consider a Kripke structure  $M$ .

$V(\text{EY}(\varphi))$	$\triangleq \widehat{A} \cdot \vec{\varphi}$	Exists Yesterday
$V(\text{ES}(\varphi, \psi))$	$\triangleq \mu \vec{z}. \vec{\psi} + (\vec{\varphi} * \widehat{A} \cdot \vec{z})$	Exists Since
$V(\text{EP}(\varphi))$	$\triangleq \mu \vec{z}. \vec{\varphi} + (\vec{1} * \widehat{A} \cdot \vec{z})$	Exists in Past
$V(\text{EH}(\varphi))$	$\triangleq \nu \vec{z}. \vec{\varphi} * \widehat{A} \cdot \vec{z}$	Exists in History
$V(\text{AY}(\varphi))$	$\triangleq \overline{\widehat{A} \cdot \vec{\varphi} - \vec{\varepsilon}}$	forAll Yesterday
$V(\text{AS}(\varphi, \psi))$	$\triangleq \nu \vec{z}. \vec{\psi} + \left( \vec{\varphi} * \left( \overline{\widehat{A} \cdot \vec{z} - \vec{\varepsilon}} \right) \right)$	forAll Since
$V(\text{AP}(\varphi))$	$\triangleq \nu \vec{z}. \vec{\varphi} + \left( \vec{1} * \left( \overline{\widehat{A} \cdot \vec{z} - \vec{\varepsilon}} \right) \right)$	forAll in Past
$V(\text{AH}(\varphi))$	$\triangleq \nu \vec{z}. \vec{\varphi} * \left( \overline{\widehat{A} \cdot \vec{z} - \vec{\varepsilon}} \right)$	forAll in History

Since the entry node does not have any predecessors, an AY formula should not hold at the entry node denoted by vector  $\vec{\varepsilon}$ . Recall that in CTL<sub>bp</sub>, past is finite. Therefore, AS is defined as the greatest fixed point. The greatest fixed point subsumes the infinite paths of loops in the Kripke structure. However, since AY does not hold at the entry node,  $\psi$  has to hold before or at the entry node. This gives the required finitary semantics to AS.

## B. Fixed points over different vector spaces

Our aim is to define correlations between temporal formulae interpreted on different Kripke structures. The Kripke structures can have different number of nodes. Some temporal operators are given fixed point semantics. Therefore, to correlate such formulae we need a correlation between fixed points of functions over vector spaces of different sizes.

Consider a complete lattice  $D_n = \langle \mathcal{B}_n, \leq_n, \perp_n, \top_n \rangle$  of boolean vectors of size  $n$  where  $\perp_n = \vec{0}_n$  and  $\top_n = \vec{1}_n$  are the top and bottom elements. Let  $D_m = \langle \mathcal{B}_m, \leq_m, \perp_m, \top_m \rangle$  be another complete lattice of boolean vectors of size  $m$ . Let  $R$  be an  $(m \times n)$  boolean matrix correlating elements of vectors in  $D_m$  and

$D_n$ . Consider monotonic functions  $f : D_n \rightarrow D_n$  and  $f' : D_m \rightarrow D_m$ . We first define an ordering between  $f$  and  $f'$  with respect to  $R$ .

**Definition 28.**  $f \sqsubseteq_R f'$  iff  $\forall \vec{z} \in D_n : R \cdot f(\vec{z}) \leq_m f'(R \cdot \vec{z})$

Recall that an  $(m \times n)$  relation  $R$  can be seen as a function  $R : D_n \rightarrow D_m$ . We now determine the correlations between the least and the greatest fixed points of functions  $f$  and  $f'$  s.t.  $f \sqsubseteq_R f'$ .

**Lemma 29.** If  $f \sqsubseteq_R f'$  then  $R \cdot (\mu \vec{z}. f(\vec{z})) \leq_m \mu \vec{z}'. f'(\vec{z}')$ .

*Proof.* By Kleene's fixed point theorem [Kle52],  $\mu \vec{z}. f(\vec{z}) \triangleq f^a(\perp_n)$  for some  $a \geq 1$  and  $\mu \vec{z}'. f'(\vec{z}') \triangleq f'^b(\perp_m)$  for some  $b \geq 1$ .

We first prove by induction that  $\forall k \in \mathcal{N} : R \cdot f^k(\perp_n) \leq_m f'^k(\perp_m)$ .

**Base case** [ $k = 1$ ]: From Definition 28,  $R \cdot f(\perp_n) \leq_m f'(R \cdot \perp_n)$ . Since  $\perp_n = \vec{0}_n$  and  $\perp_m = \vec{0}_m$ ,  $R \cdot \perp_n = \perp_m$ . Hence,  $R \cdot f(\perp_n) \leq_m f'(\perp_m)$ .

**Induction hypothesis** [ $k \geq 1$ ]: Let  $R \cdot f^k(\perp_n) \leq_m f'^k(\perp_m)$ .

**Inductive case:** From Definition 28,  $R \cdot f(f^k(\perp_n)) \leq_m f'(R \cdot f^k(\perp_n))$ . From the induction hypothesis and monotonicity of  $f'$ ,  $f'(R \cdot f^k(\perp_n)) \leq_m f'(f'^k(\perp_m)) = f'^{k+1}(\perp_m)$ . Hence, by transitivity,  $R \cdot f^{k+1}(\perp_n) \leq_m f'^{k+1}(\perp_m)$ .

Let  $\mu \vec{z}. f(\vec{z}) = f^a(\perp_n)$  and  $\mu \vec{z}'. f'(\vec{z}') = f'^b(\perp_m)$  for  $a \geq 1$  and  $b \geq 1$ . We consider all the relations between  $a$  and  $b$  and show that  $R \cdot f^a(\perp_n) \leq_m f'^b(\perp_m)$ .

**Case** [ $a < b$ ]: From Part I,  $R \cdot f^a(\perp_n) \leq_m f'^a(\perp_m)$ . Since  $f'$  is monotonic, we know that  $\forall k \in \mathcal{N} : f'^k(\perp_m) \leq_m f'^{k+1}(\perp_m)$ . Since  $a < b$ ,  $f'^a(\perp_m) \leq_m f'^b(\perp_m)$ . Hence, by transitivity,  $R \cdot f^a(\perp_n) \leq_m f'^b(\perp_m)$ .

**Case** [ $a \geq b$ ]: From Part I,  $R \cdot f^a(\perp_n) \leq_m f'^a(\perp_m)$ . If  $f'^k(\perp_m)$  is the least fixed point,  $\forall l \geq k : f'^l(\perp_m) = f'^k(\perp_m)$ . Since  $f'^b(\perp_m)$  is the least fixed point and  $a \geq b$ ,  $f'^a(\perp_m) = f'^b(\perp_m)$ . Hence, by substituting,  $R \cdot f^a(\perp_n) \leq_m f'^b(\perp_m)$ .  $\square$

Similarly, we prove the correlation between greatest fixed points stated below.

**Lemma 30.** If  $f \sqsubseteq_R f'$  then  $R \cdot (\nu \vec{z}. f(\vec{z})) \leq_m \nu \vec{z}'. f'(\vec{z}')$ .